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Trans-ionospheric signal polarization

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Abstract

An electromagnetic (EM) wave originating at or near the earth's surface can propagate upwards, traverse the ionosphere, and then be detected at a satellite. While the signal's magnitude will affect its detectabilty, its polarization will play an important role as well. Furthermore the signal polarization can reveal properties about its source, and the ionospheric conditions through which it traveled. Thus, it is desirable to predict the polarization characteristics of trans-ionospheric EM signals. This situation is complicated by the birefringence imposed on an EM wave propagating in the magnetized plasma of the ionosphere. Each frequency component of the signal can excite two waves that propagate at different phase velocities due to their separate refractive characteristics.

For this work, the amplitude and phase of each EM wave field spatial component is followed as it crosses the lower ionospheric boundary, traverses the ionosphere and then crosses the upper boundary. Results are for a single frequency, and can be easily extended to broadband waveforms using fourier transform techniques.

Contents

1 introduction

As an upward traveling EM wave encounters the underside of the ionosphere, each vector component of the wave's electric and magnetic fields will be separated into two distinct parts due to the birefringent nature of the magnetized plasma in the ionosphere [5, 3, 1, 2, 9]. Each part, or mode, may or may not propagate depending on the dispersive characteristics of the plasma at each frequency component of the incoming wave.

This mode partitioning comes from the four solutions of the magnetized Cold Plasma Dispersion Relation (CPDR): a forward and backward traveling *fast* wave root, and a forward and backward traveling *slow* wave root - fast and slow relating to their relative phase velocities. These four roots come from the solution of the wave equation derived from the Maxwell equations, which is fourth order for propagation in a magnetized plasma medium. These roots are separated into *two* principal roots (fast and slow), understanding that each has a forward and backward component.

Each mode has different dispersive characteristics, and will contribute differently to the amplitude and phase of each wave field component. Thus, in order to accurately track the wave as it propagates through the ionosphere it's field amplitudes must be parsed into fast and slow mode contributions calculated from the CPDR at the entrance and exit of the ionosphere. This information is critical in applications where EM radiation is propagated through the ionosphere and detected at a satellite-based antenna. For example, in ray tracing techniques, the amplitude and phase of the fast and slow wave modes is tracked separately through the ionosphere. Correct power partitioning between the modes is required to correctly superpose the transmitted fields at the receiver, giving the proper polarization characteristics.

The goal of this report is to accurately partition the contribution of each mode in each wave field component (or *mode* polarization) as the wave encounters, traverses, and exits the ionospheric plasma.

The ionosphere will be represented by a cold, collisionless plasma in which the ions are too massive to respond to any phenomena on the time scale of the EM wave (RF frequency).

This work draws heavily from Budden [4], chapters 1 through 8.

2 the problem

An EM pulse is generated at or near the earth's surface, with a magnitude $V(t)$ in Volts/meter. This waveform will propagate to a satellite-based detector. It must traverse three discrete regions: free space, ionosphere, and then free space to reach the detector. At each boundary with free space, the ionosphere will be represented locally by a homogeneous, anisotropic plasma; that is, it will have no spatial variation in the local plasma density or magnetic field. The boundaries between each region will be assumed abrupt (not gradual). Furthermore, given the distances involved, the waveform propagates as a far-field plane wave on either side of the ionosphere, with the electric field assuming some polarization in a plane transverse to the direction of propagation wave vector $\boldsymbol{S}$. The incident wave can be linear, elliptical, or circular. Also, due to the distances and wavelengths involved, the local geometry at the areas where the wave enters and exits the ionosphere will be assumed rectilinear.

Figure 1 shows the general situation of EM propagation through the ionosphere as it relates to the formulation in this report. A signal is assumed to originate on the earth's surface, or somewhere in the lower atmosphere. It then encounters the underside of the ionosphere and undergoes a birefringent mode splitting inside the magnetized ionospheric plasma, as well as partially reflecting from the boundary. Each mode will then propagate through the ionosphere suffering different dis-

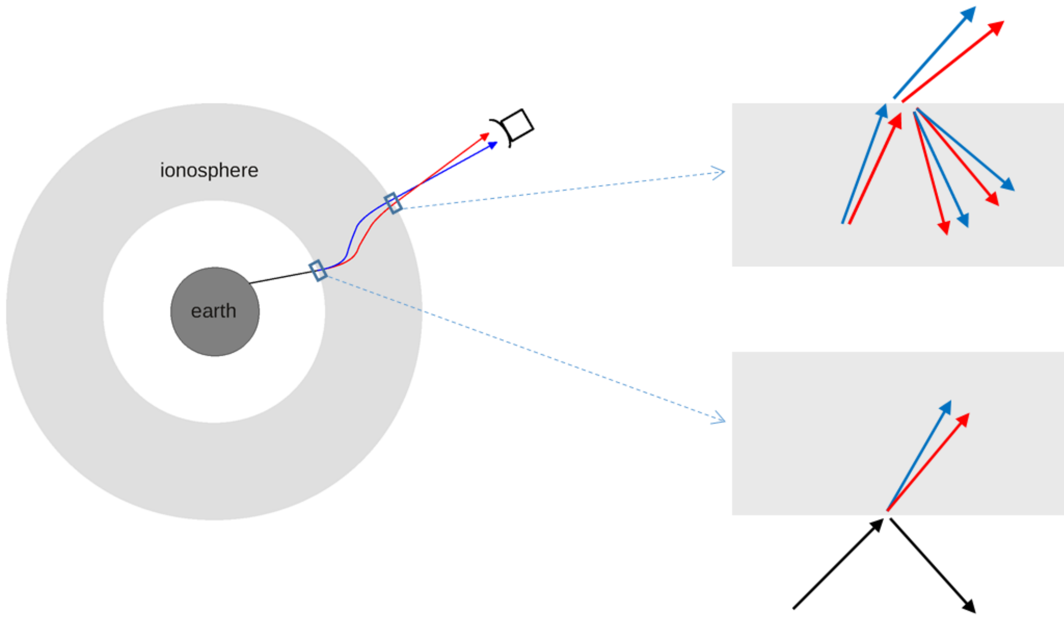

Figure 1: *EM wave propagation through the ionosphere represented by rays. The ionosphere introduces a bi-refringent mode splitting of the incident wave.*

persive characteristics (sometimes only one or neither will traverse the whole ionospheric width at a given frequency) and encounter the topside ionospheric boundary. Each mode will partially transmit and partially reflect at that boundary. The reflected part of each mode will again split, giving a total of four reflected components (multiple reflections are ignored). After crossing the topside boundary, the total electric (or magnetic) wave field can then be calculated as the superposition of each mode contribution to each wave field spatial component.

Note that for this report, internal reflections between the top and bottom of the ionosphere are not considered. This is due to the distances involved and anticipated signal strengths of the EM signals.

3 wave polarization definitions

EM waves of concern for this report have wavelengths that are minute compared to the distances they travel to be detected ($\lambda/d <<< 1$). They can therefore be treated as plane waves. By definition, the electric and magnetic field components of a plane wave in free space (a homogeneous and isotropic medium) lie in a plane transverse to the direction of propagation (wave normal). Thus, for the electric field

$$\boldsymbol{E} \perp \boldsymbol{S}$$

where $\boldsymbol{E}$ is the EM wave's electric field, $\boldsymbol{S}$ is the wave normal direction, and bold quantities represent vectors. This is not the case in a magnetized (anisotropic) plasma, where the electric field can have a component along the direction of wave propagation as well, so that in general the wave electric field can be separated into components along and across $\boldsymbol{S}$ such that

$$\boldsymbol{E} = \boldsymbol{E}_{\mathtt{em}} + \boldsymbol{E}_{\mathtt{es}}$$

where the em/es subscripts refer to electromagnetic and electrostatic components respectively according to standard definitions of wave propagation in plasmas [1, 2, 3]

$$\boldsymbol{E}_{\mathtt{em}} \perp \boldsymbol{S}, \qquad \boldsymbol{E}_{\mathtt{es}} \parallel \boldsymbol{S}$$

and $\boldsymbol{E}_{\mathtt{em}}, \boldsymbol{E}_{\mathtt{es}}$ can each be decomposed into one or more unit vectors in a given coordinate system.

The polarization state ρ of a plane wave is a quantity describing the electric field orientation $\boldsymbol{E}_{\mathtt{em}}$ at a given point in space/time as a function of time/space. For a coordinate system with $\boldsymbol{S}$ along the z axis, the polarization state is given as [4, 5]

$$\rho = E_y / E_x \tag{1}$$

where

$$E_{x,y} \;\sim\; E_{1,2}\, e^{-i(\mathfrak{n}\omega z/c - \omega t + \phi_{x,y})} \tag{2}$$

$\mathfrak{n}$ is the index of refraction, c is the speed of light, and $\phi_{x,y}$ represents any phase difference between the components that had previously existed before entering the propagation medium (say, from an antenna). In an anisotropic medium (magnetized plasma), $\mathfrak{n}$ can be different for the two field components, as will be discussed later. In general, ρ will be a complex quantity.

In a magnetized plasma, by symmetry, it is straightforward to rotate into a coordinate system to where the z axis is aligned such that $\boldsymbol{S}$ lies on the z axis, with the magnetic field $\mathbf{B_0}$ in the x - z plane at an angle θ to $\boldsymbol{S}$, as shown in Fig. 2. In this coordinate system, ρ is found in the same way as above.

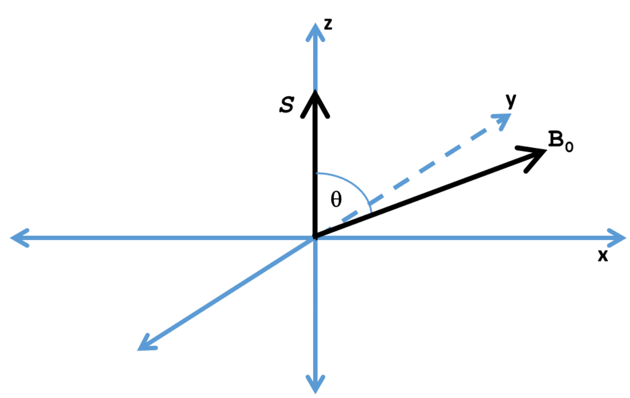

Figure 2: *Coordinate system with $\mathbf{S}$ along $\boldsymbol{z}$ and $\mathbf{B_0}$ in the $\boldsymbol{x}-\boldsymbol{z}$ plane.*

The computation of ρ is given in many standard texts in terms of the Poincaré sphere or Stokes parameters [6, 7, 8]. Only a brief summary is given here.

Referring back to equations 1 and 2, specify a fixed point in space (say $z = z_0$) to find ρ

$$\rho = \frac{E_2}{E_1}\, e^{-i\Delta\Phi} \tag{3}$$

where $\Delta\Phi$ is the phase of E_x relative to E_y

$$\Delta\Phi = \frac{\omega z_0}{c}(\mathfrak{n}_y - \mathfrak{n}_x) + \phi_y - \phi_x \tag{4}$$

Figure 3 summarizes the wave polarization states for a span of E_2/E_1 and phase differences (from Kraus [6]).

A straightforward method of detecting an electromagnetic plane wave of arbitrary polarization with a linear antenna is done by decomposing the incoming wave electromagnetic field into its individual components, and then forming the vector dot product of each with the antenna effective length (defined such that the effective length, multiplied by the amplitude in V/m of the incoming wave electric field component, equals the open circuit voltage at the terminals of the antenna [6, 8]).

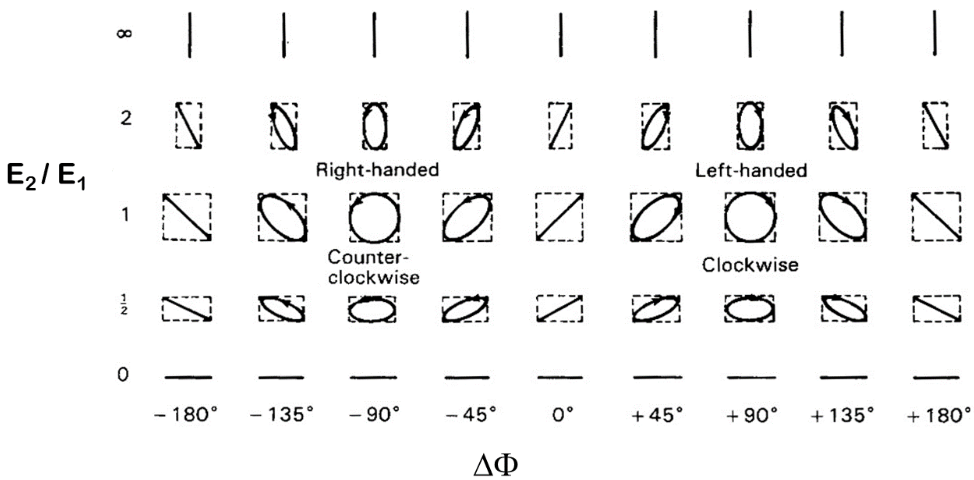

Figure 3: *Wave polarization definitions (from Kraus [6]).* E_2/E_1 *and* $\Delta\Phi$ *are the magnitude and phase of* ρ *respectively. These definitions are for a wave approaching the viewer.*

Of course, as mentioned earlier, this is all done for a single frequency. A finite bandwidth signal can be decomposed into its fourier components and each can be treated as above.

For this report, the broadband signal will travel from free space, through the ionospheric plasma, and then through free space again to reach a detector. Each frequency component of the signal will have suffered a frequency dependent phase delay for each of *two* modes in the plasma - described below. Each frequency component can be characterized as linear, circular, or elliptical; but the *total* time signal will have a complicated phase characterization determined by the individual field components, each of which has been formed by the superposition of two modes after they have emerged from the ionosphere and traveled to the antenna.

4 progressive plane waves

Any plane wave in a homogeneous medium can be decomposed into component *progressive* plane waves (PPW's). This process is similar to spatial fourier decomposition, and it will be shown in isotropic (non-magnetized) and anisotropic (magnetized) homogeneous plasma [4, 5]. Furthermore, for a magnetized plasma, the component PPW's are the two solutions to the magnetized Cold Plasma Dispersion Relation (CPDR). This construct is very helpful in calculating the mode contribution to the fields of propagating EM waves as they approach, and cross, boundaries.

A review of the constituitive relations, and their use in Maxwell's equations will help in the presentation of PPW's.

4.1 Maxwell's equations

Maxwells equations for the plane wave field components in a medium with magnetic permeability $\mu = \mu_0$ (such as a plasma) are

$$\nabla \cdot \mathbf{D} = 0 \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{7}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{8}$$

where field the field descriptions are given below -

E	Electric field intensity	(V/m)
D	Electric displacement	(Coulomb/m^2)
H	Magnetic intensity	(Amp-turn/m)
B	Magnetic induction	(T)

The electric polarization is defined as

$$\mathbf{P} \equiv Ne\mathbf{r} \quad (\text{Coulomb/m}^2) \tag{9}$$

where N is the density of electrons, e is the electron charge (a negative number), and $\mathbf{r}$ is the average length of displacement of the electron distribution; for example, the length traversed during a time period for harmonically driven displacement due to a wave field. This implies that $\mathbf{P}$ is defined over a volume that is large enough to contain many electrons given their density N. This volume has a scale length dimension of the debye length l_D [4], such that the volume is given by l_D^3. The debye length is on the order of a few centimeters or smaller in the ionosphere [4].

The electric displacement field is defined as

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \tag{10}$$

showing the contributions from the wave electric field and the polarization due to electron motion.

Consider a rectilinear geometry in which time harmonic waves, where all field quantities have time dependence of $e^{i\omega t}$, travel. The wave fields will further be assumed first order, that is, they do not change the background equilibrium of the plasma, and second order effects are negligible. Faraday's (eqn. 7) and Ampere's (eqn. 8) equations are the starting point for finding solutions for a propagating wave. Their component forms are

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega\mu_0 Hx \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\omega D_x \tag{11}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -i\omega\mu_0 Hy \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i\omega D_y \tag{12}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu_0 Hz \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega D_z \tag{13}$$

4.2 constituitive relations

In order to acquire solutions for wave **E** and **H** fields in a medium other than free space, it is necessary to relate the polarization **P** to **E** in the displacement field (eqn. 10). These relations are known as the constituitive relations for the specific medium (homogeneous plasma for this case). A homogeneous plasma can be either isotropic (non-magnetized) or anisotropic (magnetized), and each will be considered.

4.2.1 non-magnetized plasma

In this case, there is no applied magnetic field and the equation of motion for the electrons (neglecting collisions) is [3, 4, 1, 9]

$$eN\mathbf{E} = Nm_e\frac{\partial^2\mathbf{r}}{\partial t^2} \tag{14}$$

where m_e is the electron mass, and all quantities are time harmonic (first order). The definition of **P** in eq. 9 can be used to re-write eqn. 14 to give the constituitive relation between the electric field and the polarization in a non-magnetized plasma

$$\mathbf{P} \;=\; -\frac{Ne^2}{\omega^2 m_e}\mathbf{E} \;=\; -\epsilon_0 X\mathbf{E} \tag{15}$$

where X is the well known parameter in ionospheric physics

$$X \;\equiv\; \frac{Ne^2}{\omega^2\epsilon_0 m_e} \;=\; \frac{\omega_p^2}{\omega^2} \tag{16}$$

Thus, from the definition of the displacement field (eqn. 10)

$$\mathbf{D} \;=\; \epsilon_0(1-X)\mathbf{E} \;=\; \epsilon_0\epsilon_r\mathbf{E} \tag{17}$$

where ϵ_r is the relative permittivity defined from the refractive index $\mathfrak{n}$

$$\mathfrak{n}^2 \;\equiv\; \epsilon_r \;=\; 1-X \tag{18}$$

And $\mathfrak{n}$ is recognized as the refractive index for an EM wave in a non-magnetized plasma [5, 3]. It is a scalar that depends on frequency and plasma density. Note that the relationship between *each spatial component* of **D** and **E** for this case is proportional

$$\mathbf{D} = \epsilon_0\mathfrak{n}^2\mathbf{E} \quad\Longrightarrow\quad \begin{array}{l} D_x \propto E_x \\ D_y \propto E_y \\ D_z \propto E_z \end{array} \tag{19}$$

4.2.2 magnetized plasma

Here, the homogeneous plasma is immersed in a static magnetic field $\mathbf{B}_0$. The equation of motion for electrons is [3, 5, 1, 2, 9]

$$eN\mathbf{E} + eN\frac{\partial\mathbf{r}}{\partial t}\times\mathbf{B}_0 = Nm_e\frac{\partial^2\mathbf{r}}{\partial t^2} \tag{20}$$

and all wave quantities ($\mathbf{E}$, $\mathbf{r}$) are time harmonic. Again, using the definition of $\mathbf{P}$ from eqn. 9, this is re written as

$$\frac{Ne^2}{m_e\omega^2}\mathbf{E}+\frac{ie}{m\omega}\mathbf{P}\times\mathbf{B}_0=\mathbf{P} \tag{21}$$

Now introduce the vector quantity

$$\mathbf{Y}=\frac{e\mathbf{B}_0}{m\omega}=\frac{-\omega_c}{\omega}\left(l_x\hat{x}+l_y\hat{y}+l_z\hat{z}\right) \tag{22}$$

which is also well known in the ionospheric physics field. The direction cosines of $\mathbf{B_0}$ are l_x, l_y, l_z in the rectilinear coordinate system. Also note that the electron charge e is negative, so that the vector $\mathbf{Y}$ will be antiparallel to $\mathbf{B_0}$. Equation 21 is now

$$-\epsilon_0 X\mathbf{E} \;=\; \mathbf{P}+i\mathbf{P}\times\mathbf{Y} \;=\; \overline{\mu}\cdot\mathbf{P} \;=\; \begin{bmatrix} 1 & iYl_z & -iYl_y \\ -iYl_z & 1 & iYl_x \\ iYl_y & -iYl_x & 1 \end{bmatrix}\cdot\mathbf{P} \tag{23}$$

where eqn. 16 has been used. This is the constituitive relation between $\mathbf{P}$ and $\mathbf{E}$ in a magnetized plasma.

Using this result and eqn. 10, the displacement field is

$$\mathbf{D} \;=\; \epsilon_0\mathbf{E}-\epsilon_0 X\,\overline{\mu}^{\,-1}\cdot\mathbf{E} \;=\; \epsilon_0\left(\bar{\mathbf{I}}-X\,\overline{\mu}^{\,-1}\right)\mathbf{E} \tag{24}$$

where the tensor $\bar{\mathbf{I}}$ is

$$\bar{\mathbf{I}} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The refractive index is now a tensor as well

$$\mathfrak{n} \;=\; \bar{\mathfrak{n}} \;=\; \left(\bar{\mathbf{I}}-X\,\overline{\mu}^{\,-1}\right)^{1/2} \tag{25}$$

Note now that a single spatial component of $\mathbf{D}$ depends on *all three* spatial components of $\mathbf{E}$.

$$\begin{aligned} D_x &\propto E_x, E_y, E_z \\ D_y &\propto E_x, E_y, E_z \\ D_z &\propto E_x, E_y, E_z \end{aligned} \tag{26}$$

4.3 plane and progressive plane wave solutions to Maxwell's equations

As mentioned earlier, any arbitrary plane wave can be decomposed into component progressive plane waves. In what follows, the plane wave solutions to Maxwell's equations will be found for both the isotropic and anisotropic homogeneous plasma cases. These solutions will be shown to consist of component progressive plane waves: a situation similar to fourier decomposition in space.

A plane wave is defined to be a "disturbance in which there is no variation of any field component in any plane parallel to a fixed plane" [4]. For simplicity, choose the z axis to be normal to this $(x-y)$ plane. This axis is then said to be the *wave normal* direction. Thus, for all field components in the plane wave

$$\frac{\partial}{\partial x},\frac{\partial}{\partial y} \;\rightarrow\; 0 \tag{27}$$

Ampere's and Faraday's equations (eqns. 11 - 13) are then

$$\frac{\partial E_y}{\partial z} = i\omega\mu_0 Hx \qquad \frac{\partial H_y}{\partial z} = -i\omega D_x \tag{28}$$

$$\frac{\partial E_x}{\partial z} = -i\omega\mu_0 Hy \qquad \frac{\partial H_x}{\partial z} = i\omega D_y \tag{29}$$

$$H_z = 0 \qquad D_z = 0 \tag{30}$$

showing that the **D** and **H** fields are perpendicular (also called transverse)to the plane wave normal.

4.3.1 non-magnetized plasma

It has already been shown in section 4.2.2 that each spatial component of **D** is directly proportional to the corresponding spatial component of **E**, and thus **D**, **H**, and **E** are all transverse and no field component exists in the wave normal (z) direction. From eqns. 17, 28, and 29, two *independent* wave equations for E_x and E_y can be derived

$$\frac{\partial^2 E_x}{\partial z^2} + k^2\mathfrak{n}^2 E_x = 0 \qquad \frac{\partial^2 E_y}{\partial z^2} + k^2\mathfrak{n}^2 E_y = 0 \tag{31}$$

where $k = \omega/c = \sqrt{\epsilon_0\mu_0}$ and the index of refraction is $\sqrt{1-X}$ from eqn. 18. Two important items must be pointed out. First, both waves see the same refractive index. Second, the solutions for E_x and E_y are independent, that is, one can change or even vanish without affecting the other. This means that each wave is linearly polarized; one in x, and one in y. The solutions to eqn. 31 are

$$E_x^{(1)} e^{-ik\mathfrak{n}z} \qquad E_x^{(2)} e^{+ik\mathfrak{n}z} \tag{32}$$

$$E_y^{(1)} e^{-ik\mathfrak{n}z} \qquad E_y^{(2)} e^{+ik\mathfrak{n}z} \tag{33}$$

where $E_x^{(1),(2)}$, $E_y^{(1),(2)}$ are independent of z and t, as well as each other. These four solutions represent two forward ($+z$) and two backward ($-z$) traveling plane waves that can propagate in a homogeneous, isotropic medium.

Consider the forward traveling x-polarized wave. Substitution of the left side eqn. of 32 into Faraday's law (left eqn. of 29) gives

$$\frac{E_x^{(1)}}{H_y^{(1)}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{\mathfrak{n}} = \mathfrak{Z} = \mathfrak{Z}_0\frac{1}{\mathfrak{n}} \tag{34}$$

This ratio is known as the wave impedance of the forward traveling plane wave. The parameter $\mathfrak{Z}_0$ is approximately 377 Ω [4], so that in free-space, where $\mathfrak{n} = 1$, this wave's impedance will be $\mathfrak{Z}_0$. In a homogeneous isotropic medium, $\mathfrak{n}$ is a scalar value, and thus $\mathfrak{Z}$ is a constant scalar with no spatial dependence. Furthermore, the field quantities E_x and H_y of this wave depend on the wave normal direction (z) only through the factor $exp(-ik\mathfrak{n}z)$. **A plane wave with a spatially independent wave impedance whose field components dependence on the wave normal direction (z for this case) is given by $exp(-ik\mathfrak{n}z)$ is called a *progressive plane wave*** [4].

Similarly, for the $-z$ traveling x-pol and $\pm z$ traveling y-pol waves in eqns. 32 and 33

$$\frac{E_x^{(2)}}{H_y^{(2)}} = -\mathfrak{Z}_0\frac{1}{\mathfrak{n}}, \qquad \frac{E_y^{(1)}}{H_x^{(1)}} = -\mathfrak{Z}_0\frac{1}{\mathfrak{n}}, \qquad \frac{E_y^{(2)}}{H_x^{(2)}} = +\mathfrak{Z}_0\frac{1}{\mathfrak{n}} \tag{35}$$

showing that they are also progressive plane waves.

Consider a different solution to the x-pol plane wave equation (31) in the forward z direction

$$E_x = E_x^{(1)} \cos(k\mathfrak{n}z) \tag{36}$$

This is a plane wave, and is easily decomposed into two component PPW's from eqn. 32. However, substitution of this solution into Faraday's equation (29) gives

$$\frac{E_x^{(1)}}{H_y^{(1)}} = i\mathfrak{Z}_0 \frac{1}{\sin(k\mathfrak{n}z)} \tag{37}$$

which shows that this plane wave is not a PPW because the wave impedance is not independent of the wave normal direction.

The examples given above were for an x-pol plane wave, and apply equally to a y-pol plane wave. Any arbitrarily $x-y$ polarized plane wave can be decomposed into x and y components found from eqns. 32 and 33. Furthermore, the plane waves considered above were traveling in the $\pm z$ direction. Naturally, the results apply for an arbitrary wave normal direction, where the fields are specified in a plane perpendicular to that direction. In that case, a combination of both x and y polarized PPW's can be used to construct an arbitrary plane wave.

To summarize, in a homogeneous, isotropic medium an arbitrary plane wave can be decomposed into linearly polarized PPW's where the E-field of one will be perpendicular to the E-field of the other. There are no restrictions on the amplitude or relative phasing of the component PPW's.

Mathematically, solutions to the wave equation derived from Faraday's and Ampere's laws for plane waves and propagating plane waves can be summarized as follows for $\pm z$ directed propagation

$$\text{PW}: \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \;=\; 0 \qquad\qquad \text{PPW}: \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \;=\; 0; \quad \frac{\partial}{\partial z} \;=\; \pm\, i k\mathfrak{n} \tag{38}$$

or, more generally

$$\text{PW}: \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \;=\; 0 \qquad\qquad \text{PPW}: \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \;=\; 0; \quad \frac{\partial}{\partial x_3} \;=\; \pm\, i k\mathfrak{n} \tag{39}$$

for a PW or PPW traveling in the $\pm x_3$ direction, where the x_1, x_2, x_3 right handed coordinate system is rotated relative to the x, y, z coordinate system in order to make the wave normal direction parallel to a principal axis.

4.3.2 the notation H and $\mathscr{H}$

In free-space, the PPW wave impedance is $\mathfrak{Z}_0$. From now on, the magnetic intensity **H** will be replaced with

$$\mathscr{H} = \mathfrak{Z}_0 \mathbf{H} \tag{40}$$

This measures the magnetic field in terms of the electric field that would be associated with it in free-space [4].

This also helps simplify frequently used equations. For example, Faraday's and Ampere's equations for a plane wave traveling in the $\pm z$ direction (28, 29, 30) are now

$$\frac{\partial E_y}{\partial z} = ik\mathscr{H}_x \qquad\qquad \frac{\partial \mathscr{H}_y}{\partial z} = \frac{-ik}{\epsilon_0} D_x \tag{41}$$

$$\frac{\partial E_x}{\partial z} = -ik\mathscr{H}_y \qquad\qquad \frac{\partial \mathscr{H}_x}{\partial z} = \frac{ik}{\epsilon_0} D_y \tag{42}$$

$$\mathscr{H}_z = 0 \qquad\qquad D_z = 0 \tag{43}$$

Also, the wave impedance relationships are simplified. For example, eqn. 34 is now

$$\frac{E_x}{\mathscr{H}_y} = \frac{1}{\mathfrak{n}} \tag{44}$$

4.3.3 magnetized plasma

PPW solutions to Faraday's and Ampere's eqns. 41, 42, and 43, using the results from eqn. 38 give

$$\mathfrak{n}E_y = -\mathscr{H}_x \qquad \mathfrak{n}\mathscr{H}_y = \frac{1}{\epsilon_0}D_x \tag{45}$$

$$\mathfrak{n}E_x = \mathscr{H}_y \qquad \mathfrak{n}\mathscr{H}_x = -\frac{1}{\epsilon_0}D_y \tag{46}$$

$$\mathscr{H}_z = 0 \qquad D_z = 0 \tag{47}$$

However, from section 4.2.2 the spatial components of **D** each depend on all three spatial components of **E** because $\mathfrak{n}$ is a tensor. Hence, the wave equation for E_y derived from the left equation of 45 and the right equation of 46 is *not* independent of the wave equation for E_x derived from the right equation of 45 and the left equation of 46. This means that the polarization states of the PPW solutions will be restricted.

Instead of proceeding to calculate each wave equation, the characteristics of PPW's in a magnetized plasma are more clearly understood in terms of their individual polarization states. Consider a $+z$ directed PPW in a magnetized plasma. By symmetry, the x and y axes can be rotated about the z axis such that the magnetic field, and hence the vector **Y** lies in the $x-z$ plane at an angle θ to the z axis. The direction cosines for **Y** are then

$$l_x = \sin\theta, \qquad l_y = 0, \qquad l_z = \cos\theta \tag{48}$$

Under the PPW assumption, eqn. 47, the z component of the displacement is

$$D_z = \epsilon_0 E_z + P_z = 0 \tag{49}$$

The z component of E is found from the constituitive relation, eqn. 23, and eqn. 49 becomes

$$(1-X)P_z = iY\sin\theta P_y \tag{50}$$

where $Y = |\mathbf{Y}|$. The polarization state of a z directed PPW can be written in several equivalent forms using Faraday's and Ampere's equations along with the definition of the displacement

$$\rho = \frac{E_y}{E_x} = -\frac{\mathscr{H}_x}{\mathscr{H}_y} = \frac{D_y}{D_x} = \frac{P_y}{P_x} \tag{51}$$

To calculate ρ, divide the y component by the x component of eqn. 23 and use eqns. 51 and 50 [5]

$$\rho = \frac{\frac{1}{2}iY\sin^2\theta \pm \left[\frac{1}{4}Y^2\sin^4\theta + \cos^2\theta(1-X)^2\right]^{1/2}}{(1-X)\cos\theta} \tag{52}$$

Showing that only *two* PPW's can exist, with polarization states defined above. Thus, any plane wave in a magnetized plasma with an arbitrary polarization state can be resolved into two PPW's with polarizations specified by eqn. 52 with different relative phases and magnitudes. However, unlike the non-magnetized plasma PPW's, these PPW's will have *different* refractive indices, where each one is a root of the magnetized plasma CPDR. To show this, start by eliminating $\mathscr{H}_x$ and $\mathscr{H}_y$ in eqns. 45 and 46

$$D_x = \epsilon_0 \mathfrak{n}^2 E_x \qquad D_y = \epsilon_0 \mathfrak{n}^2 E_y \tag{53}$$

and these are the same as for a non-magnetized plasma. However, $E_z = 0$ for PPW's in a non-magnetized plasma (from eqn. 30 and 17) whereas $E_z \neq 0$ for PPW's in a magnetized plasma (from eqn. 30 and 26). Use D_x, and D_y from eqn. 53 in the displacement field eqn. 10 to get

$$P_x = \epsilon_0(\mathfrak{n}^2 - 1)E_x \qquad P_y = \epsilon_0(\mathfrak{n}^2 - 1)E_y \tag{54}$$

Now substitute these into the x component of eqn. 23 and divide that by P_x using $\rho = P_y/P_x$ [5, 4]. This results in the expression

$$\frac{X}{\mathfrak{n}^2 - 1} = -1 - i\rho Y \cos\theta \tag{55}$$

upon re-arranging, the equation for the refractive index becomes

$$\mathfrak{n}^2 = 1 - \frac{X}{1 + i\rho Y \cos\theta} \tag{56}$$

Substituting the expression for ρ from eqn. 52, the equation for the refractive index is

$$\mathfrak{n}^2 = 1 - \frac{X(1-X)}{1 - X - \frac{1}{2}Y^2\sin^2\theta \pm \left[\frac{1}{4}Y^4\sin^4\theta + Y^2\cos^2\theta(1-X)^2\right]^{1/2}} \tag{57}$$

which is the well known Appleton-Hartree magnetized CPDR [4, 5, 1, 9]. For reference, if electron collisions are included in the derivation, the Appleton-Hartree magnetized CPDR is

$$\mathfrak{n}^2 = 1 - \frac{X}{1 - iZ - \dfrac{\frac{1}{2}Y^2\sin^2\theta}{1 - X - iZ} \pm \left[\dfrac{\frac{1}{4}Y^4\sin^4\theta}{(1-X-iZ)^2} + Y^2\cos^2\theta\right]^{1/2}} \tag{58}$$

where the quantity Z includes the collision frequency for electrons, ν, and the wave frequency

$$Z = \frac{\nu}{\omega} \tag{59}$$

4.3.4 magnetized plasma PPW polarization relations

Eqn. 55 can be rearranged to give the expression for ρ corresponding to each PPW in a magnetized plasma

$$\rho = \frac{E_y}{E_x} = \frac{1 + \dfrac{X}{\mathfrak{n}^2 - 1}}{-iY\cos\theta} \tag{60}$$

where now $\mathfrak{n}$ takes one of its two values specified in Eqn. 57 (for a collisionless plasma). A nonzero wave electric field component in the direction of propagation can also exist in a plasma. This field can also be specified in a polarization ratio expression, say Q. Take the z-component of eqn. 23 (recall that $\mathbf{B_0}$ lies in the $x - z$ plane such that $l_y = 0$). From this equation, the expressions for P_x, P_y, and P_z are substituted using the definition of the displacement field, eqn. 10, and equations 54 and 49 respectively. This gives

$$Q = \frac{E_z}{E_y} = \frac{-iY\sin\theta(\mathfrak{n}^2 - 1)}{1 - X} \tag{61}$$

4.4 summary

Any plane wave in a homogeneous medium can be decomposed into component PPW's.

In a homogeneous, isotropic medium an arbitrary plane wave can be decomposed into two linearly polarized PPW's where the E-field of one will be perpendicular to, and independent of, the E-field of the other. Each wave sees the same refractive index.

In a homogeneous, non-isotropic magnetized plasma medium, the two component PPW's are the two solutions to the magnetized CPDR. Each wave's polarization state will be given by one of the forms of eqn. 52. Furthermore, each wave will see a different refractive index.

The usefulness of PPWs in magnetized plasma is now evident in that the mode contribution to an arbitrary propagating plane wave is found by resolving it into its component PPWs. Also, PPW's see a constant wave impedance $\mathfrak{Z}$ in their direction of travel. This is very helpful when matching E and $\mathscr{H}$ fields across boundaries according to Snell's law using Ampere's and Faraday's equations 45, 46, and 47.

5 wave incidence at the bottom of the ionosphere

The initial wave will have field components relative to a given coordinate system. The relative strength and phase of each component in the reference coordinate system will define the wave's initial polarization. However, once the field is incident on the ionospheric underside, a judicious coordinate transformation will help in specifying the electric field polarization as it enters the ionosphere. This transform will have a principal axis normal to the vacuum/plasma boundary.

The magnetized plasma in the ionosphere will decompose each frequency component into two modes, and each of those components will suffer different dispersive effects within each mode. It is therefore advantageous to work in the frequency domain and transform back to the time domain once the fields have been solved.

Consider an EM plane wave incident at the bottom of the ionosphere. It is necessary to find the portion of the wave fields that cross the boundary as well as reflect back. Furthermore, it is required to determine the portion of each spatial component of the incident field that goes into each of the two propagating modes in a magnetized plasma.

The vacuum/plasma boundary will be treated in a local rectilinear geometry. Coordinate rotations at the point of interface serve to greatly simplify calculation of the wave field's propagation across the boundary. In order to better illustrate the situation, a rotation where the incident wave and magnetic field are co-planar with a principle axis plane will be introduced. From there, the general case of oblique wave incidence and magnetic field orientation will be solved.

The advantages of this approach are due to the fact that a propagating ($\mathfrak{n}$ pure real) EM wave will refract in the magnetic meridian plane [4, 9], that is, the plane formed by the magnetic field (or vector **Y**) and the incident wave normal **S** in a *collisionless*, homogeneous, magnetized plasma.

This treatment of the vacuum/ionosphere boundary is equivalent to assuming a plane stratification of vacuum/plasma/vacuum. A tool of particular importance to this type of problem is the Booker quartic [5, 4, 10, 11, 12], which will be introduced first.

5.1 plane stratified media and the Booker quartic

A magnetized plasma occupies the volume $z \geq 0$ as shown in fig. 4. An EM plane wave **S** is incident on this plasma half-space from below ($z < 0$) and makes an angle θ_i with the z axis in $x - z$ plane.

The magnetic field vector **Y** is completely oblique with respect to the x, y, and z axes, and is defined according to its direction cosines l, m, n as

$$\mathbf{Y} = Y\left(l\hat{x} + m\hat{y} + n\hat{z}\right) \tag{62}$$

where

$$l = \cos\alpha \quad m = \cos\beta \quad n = \cos\gamma \tag{63}$$

and $Y = |\mathbf{Y}|$. Likewise, **S** is defined as

$$\mathbf{S} = S\left(\sin\theta_i\hat{x} + 0\hat{y} + \cos\theta_i\hat{z}\right) \tag{64}$$

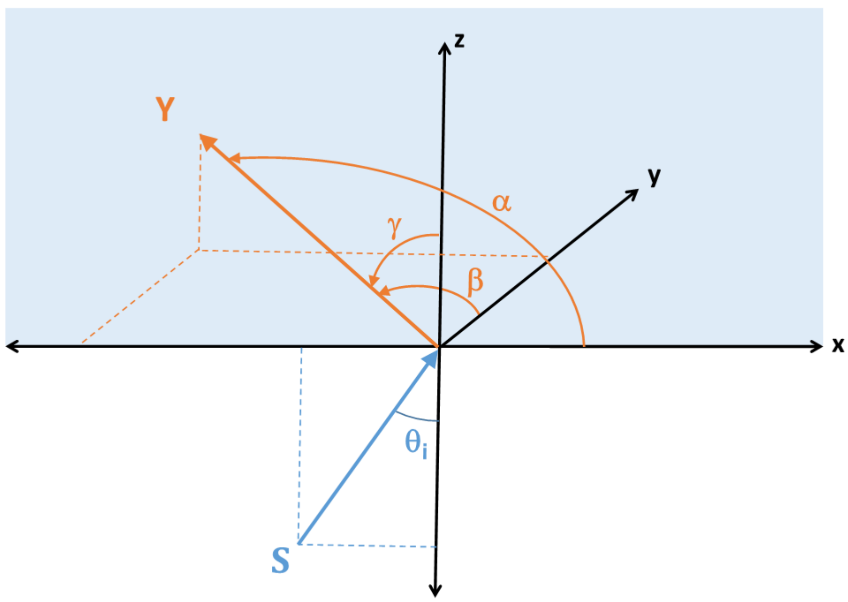

Figure 4: *Geometry for a plane wave* **S** *incident from below on a plasma half-space defined by the volume* $z \geq 0$*.* **S** *is in the* $x - z$ *plane and* **Y** *is oblique.*

Part of **S** will be transmitted across the plasma boundary and refract into a fast and slow mode, say $\mathbf{S_a}$ and $\mathbf{S_b}$. These two waves will refract in the $\mathbf{S}-\mathbf{Y}$, or magnetic meridian, plane at angles θ_a and θ_b with respect to the vertical (which is still the $\hat{z}$ axis in the $\mathbf{S}-\mathbf{Y}$ plane), and will see refractive indices $\mathfrak{n}_a$ and $\mathfrak{n}_b$. The refractive index on the vacuum side is 1.

Snell's law for the incident wave and *either* of the refracted waves is

$$\sin\theta_i = \mathfrak{n}_a\sin\theta_a = \mathfrak{n}_b\sin\theta_b \tag{65}$$

In what follows, the derivation applies to each of the refracted waves, so the a, b subscripts will be dropped and only one considered

$$\sin\theta_i = \mathfrak{n}\sin\theta \tag{66}$$

Both $\mathfrak{n}$ and θ are unknown, but the product $\mathfrak{n}\sin\theta$ is known from eqn. 66. Let

$$q = \mathfrak{n}\cos\theta \tag{67}$$

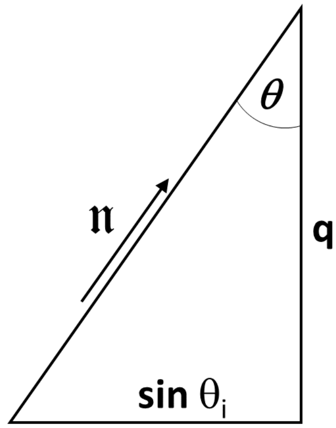

Figure 5: *Graphical representation Snell's law as it applies to the relationship between* $\mathfrak{n}$*,* q*, the refraction angle* θ*, and the incident angle* θ_i*.*

and consider $\mathfrak{n}$ as a vector (of magnitude $|\mathfrak{n}|$) inclined at an angle θ to the vertical with a horizontal component $\sin\theta_i$ and a vertical component q, as shown in fig. 5.

It is clear that if q can be found, then $\mathfrak{n}$ and θ are known from geometrical relationships

$$\mathfrak{n}^2 = q^2 + \sin^2\theta_i \qquad \tan\theta = \frac{\sin\theta_i}{q} \tag{68}$$

and therefore the task at hand is to solve for q. For convenience, let

$$S = \sin\theta_i \quad C = \cos\theta_i \tag{69}$$

where S is not associated with the wave normal vector $\mathbf{S}$. Inside the ionosphere, the refracted wave will have direction cosines $(\sin\theta,\ 0,\ \cos\theta)$ in the $\mathbf{S}-\mathbf{Y}$ plane. Using equations 68 and fig. 5, these direction cosines can be written as

$$\frac{S}{\sqrt{q^2+S^2}}, \quad 0, \quad \frac{q}{\sqrt{q^2+S^2}} \tag{70}$$

The cosine of the angle between the refracted wave in the plasma and the vector $\mathbf{Y}$ is, from eqns. 62 and 70,

$$\cos\Psi = \frac{lS+qn}{\sqrt{q^2+S^2}} \tag{71}$$

(recall that n is the direction cosine of $\mathbf{Y}$ with the z axis, and $\mathfrak{n}$ is the refractive index). Thus the component of $\mathbf{Y}$ in the direction of the refracted wave normal is

$$Y_L = Y\frac{lS+qn}{\sqrt{q^2+S^2}} \tag{72}$$

Eqns. 68 and 69 show that

$$\mathfrak{n}^2 - 1 = q^2 - C^2 \tag{73}$$

From this, the Appleton-Hartree magnetized CPDR, eqn. 58, can be written as

$$U - \frac{\frac{1}{2}Y_T^2}{U-X} + \frac{X}{q^2-C^2} = \sqrt{\frac{\frac{1}{4}Y_T^4}{(U-X)^2} + Y_L^2} \tag{74}$$

where $U = 1 - iZ$, $Y_T = Y\sin\theta$, and$Y_L = Y\cos\theta$. If both sides of this equation are squared, the term $\frac{1}{4}Y_T^2 \ / \ (U-X)^2$ is subtracted out, the resulting equation is multiplied by $U - X$, and the relation $Y_T^2 = Y^2 - Y_L^2$ along with eqn. 72 is used, the result is a quadratic equation in q [5, 4]

$$F(q) \equiv \alpha q^4 + \beta q^3 + \gamma q^2 + \delta q = \epsilon = 0 \tag{75}$$

where

$$\alpha = U(U^2 - Y^2) + X(n^2Y^2 - U^2) \tag{76}$$
$$\beta = 2lnSXY^2 \tag{77}$$
$$\gamma = -2U(U-X)(C^2U - X) + 2Y^2(C^2U - X) + XY^2(1 - C^2n^2 + S^2l^2) \tag{78}$$
$$\delta = -2C^2lnSXY^2 \tag{79}$$
$$\epsilon = (U-X)(C^2U^2 - X)^2 - C^2Y^2(C^2U - X) - l^2S^2C^2XY^2 \tag{80}$$

This equation is known as the Booker quartic [10, 11, 4, 5]. It gives four values of q, two upgoing (up from the $x-y$ plane with a $+z$ component), and two downgoing waves (down from the $x-y$ plane with a -z component). The two upgoing waves will be the two refracted waves in the plasma, and using eqn. 68, they will give $\mathfrak{n}_{a,b}$ and $\theta_{a,b}$. Collisional effects are not included in this report ($Z \to 0, U \to 1$); therefore, to find the solutions of q that correspond to up- or downgoing waves, give the Z term (eqn, 59) a very small non-zero value. The solutions for q that have a negative imaginary part will correspond to upgoing waves [4]. Note that for vertical incidence, $S = 0$ and $C = 1$, $q^2 = \mathfrak{n}^2$ and the Booker quartic reduces to the Appleton-Hartree equation (eqn. 57, or eqn. 58 if collisions are included). It should be evident by now that the Booker quartic is another form of the magnetized CPDR, but one that includes the restrictions of Snell's law for stratified media. It is extremely helpful for solving wave fields in situations that involve a stratification, and will be used throughout this report.

5.2 case I: magnetic meridian plane along principal axes

Here, the incident wave and magnetic field will both be constrained to the $x-z$ plane, as shown in fig. 6

In this case, the vacuum/plasma interface is the $x-y$ plane, the plasma volume is the region $z \geq 0$, and a plane wave $\mathbf{S_i}$ is incident from below at an angle θ_i. It reflects at an angle θ_r and, due to the bi-refringent nature of the magnetized plasma, refracts as $\mathbf{S_a}$ and $\mathbf{S_b}$ at angles θ_a and θ_b respectively. Wave field orientation will be referenced according to the wave normal direction for all waves ($\mathbf{S_i}$, $\mathbf{S_r}$, $\mathbf{S_a}$, and $\mathbf{S_b}$). Direction cosines for the vectors are

$$\mathbf{Y} \quad (l,\ 0,\ n) \tag{81}$$
$$\mathbf{S_i} \quad (\sin\theta_i,\ 0,\ \cos\theta_i) \tag{82}$$
$$\mathbf{S_r} \quad (\sin\theta_r,\ 0,\ \cos\theta_r) = (\sin\theta_i,\ 0,\ -\cos\theta_i) \tag{83}$$
$$\mathbf{S_a} \quad (\sin\theta_a,\ 0,\ \cos\theta_a) \tag{84}$$
$$\mathbf{S_b} \quad (\sin\theta_b,\ 0,\ \cos\theta_b) \tag{85}$$

where $\theta_i = \pi - \theta_r$ from Snell's law, and equations 62 and 63 were used in equation 81.

Wave field components will be either along, or perpendicular to, the wave normal direction. The wave field components along the wave normal direction are assigned the subscript L. Those perpendicular to the wave normal direction are further catagorized as parallel to the magnetic meridian

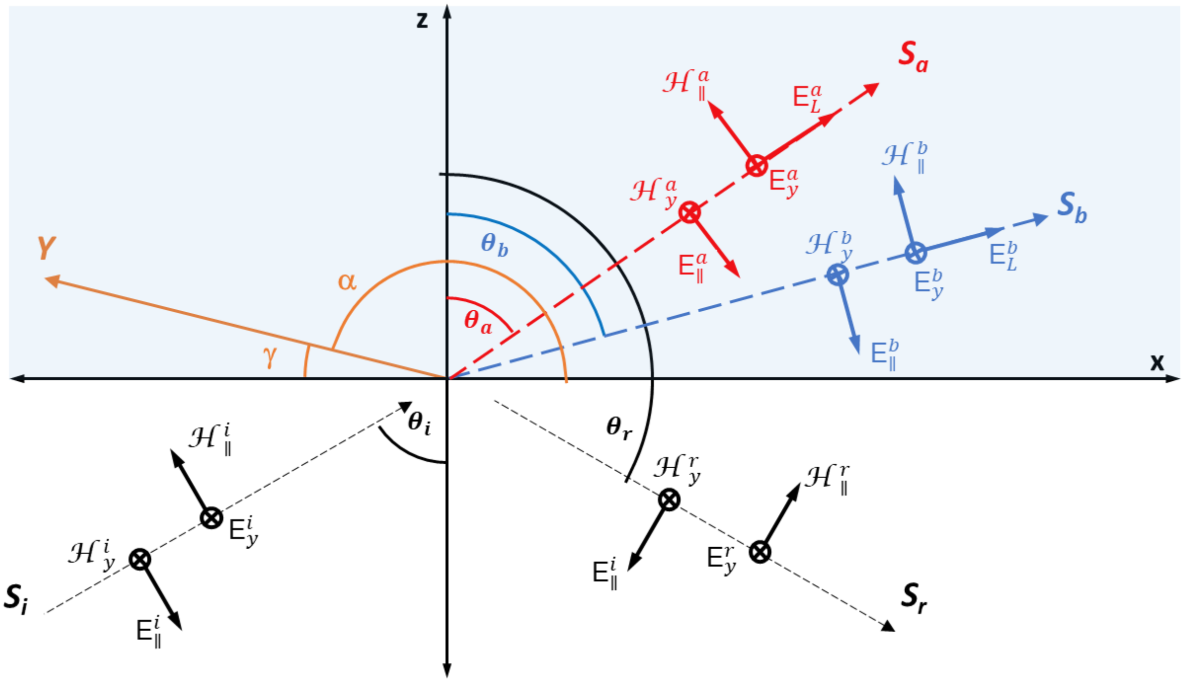

Figure 6: *Geometry for wave and magnetic field co-planar. Note that the y-coordinate points in to the page, and is different from the vector* $\mathbf{Y}$ *(in orange). The plasma volume is defined by $z \geq 0$. The vectors* $\mathbf{Y}$, $\mathbf{S_i}$, *and* $\mathbf{S_r}$ *have direction cosines* $(\cos\alpha, 0, \cos\gamma)$, $(\sin\theta_i, 0, \cos\theta_i)$, *and* $(\sin\theta_r, 0, \cos\theta_r)$*; where, by Snell's law, $\pi - \theta_r = \theta_i$. The vectors* $\mathbf{S_a}$ *and* $\mathbf{S_b}$ *have direction cosines* $(\sin\theta_a, 0, \cos\theta_a)$ *and* $(\sin\theta_b, 0, \cos\theta_b)$.

$(x-z)$ plane with a subscript $\parallel$, or perpendicular to that plane in the y direction with a y subscript. For example, from eqn. 1, the polarization state of the incident wave in Fig. 6 is

$$\rho = \frac{E_y^i}{E_{\parallel}^i} \tag{86}$$

and the relationship between the E and $\mathscr{H}$ components is given by Faraday's law, on the left side of eqns. 45 and 46,

$$\frac{E_{\parallel}^i}{\mathscr{H}_y^i} = -\frac{E_y^i}{\mathscr{H}_{\parallel}^i} = \frac{1}{\mathfrak{n}} \tag{87}$$

where, in vacuum, $\mathfrak{n} = 1$. Also note that in vacuum, by definition, a plane wave will not have an electrostatic, or longitudinal, component. Therefore, there is no E_L component in either the incident or reflected waves, while one does exist for the two refracted waves in the plasma.

With the above assumptions, the incident, reflected, and transmitted fields can now be characterized using the coordinate system in fig. 6.

5.2.1 incident fields

The components of the incident wave fields are

$$\mathscr{H}^i_x = -\mathscr{H}^i_\parallel \cos\theta_i = -E^i_y \cos\theta_i \qquad E^i_x = E^i_\parallel \cos\theta_i \tag{88}$$

$$\mathscr{H}^i_y = \mathscr{H}^i_y = E^i_\parallel \qquad E^i_y = E^i_y \tag{89}$$

$$\mathscr{H}^i_z = \mathscr{H}^i_\parallel \sin\theta_i = -E^i_y \sin\theta_i \qquad E^i_z = -E^i_\parallel \sin\theta_i \tag{90}$$

where Faraday's law, eqn. 87, was used in vacuum ($\mathfrak{n} = 1$) to relate the $\mathscr{H}$ and E fields.

5.2.2 reflected fields

The components of the reflected wave fields are

$$\mathscr{H}^r_x = \mathscr{H}^r_\parallel \cos\theta_i = E^r_y \cos\theta_i \qquad E^r_x = -E^r_\parallel \cos\theta_i \tag{91}$$

$$\mathscr{H}^r_y = \mathscr{H}^r_y = E^r_\parallel \qquad E^r_y = E^r_y \tag{92}$$

$$\mathscr{H}^r_z = \mathscr{H}^r_\parallel \sin\theta_i = -E^r_y \sin\theta_i \qquad E^r_z = -E^r_\parallel \sin\theta_i \tag{93}$$

where again Faraday's law, eqn. 87, was used in vacuum ($\mathfrak{n} = 1$) to relate the $\mathscr{H}$ and E fields, and $\theta_r = \pi - \theta_i$ from Snell's law.

5.2.3 transmitted fields

Above the vacuum/plasma interface, the wave will refract into fast and slow mode components - say mode a and mode b. Each mode will have spatial field components that must be accounted for, as in fig. 6. There will also be longitudinal (along the wave normal direction) E-field components in the plasma. The polarization relations, equations 60 and 61, for the parallel and longitudinal (magnetic meridian plane) E-field components can be used to relate them to the E_y component *for each mode*.

For this geometry, where the plasma wave is oblique in the magnetic meridian plane instead of traveling only in the z-direction, equations 60 and 61 are re-written as

$$\rho_j = \frac{E^j_y}{E^j_\parallel} = \frac{1 + \dfrac{X}{\mathfrak{n}^2_j - 1}}{-iY^j_L} \qquad Q_j = \frac{E^j_L}{E^j_y} = \frac{-iY^j_T(\mathfrak{n}^2_j - 1)}{1 - X} \tag{94}$$

where Y_L^j and Y_T^j are the components of $\mathbf{Y}$ parallel and perpendicular to the wave normal $\mathbf{S}_j$ respectively

$$Y_L^j = Y\left(l\sin\theta_j + n\cos\theta_j\right) \tag{95}$$

$$Y_T^j = Y\left(l\cos\theta_j - n\sin\theta_j\right) \tag{96}$$

$\mathfrak{n}_j$ and θ_j are the index of refraction and refractive angle found from the Booker quartic for an upward propagating wave, and $j = a, b$.

With these definitions, the transmitted fields for each mode are

$$\begin{aligned} \mathscr{H}_x^j &= -\mathscr{H}_\parallel^j \cos\theta_i & E_x^j &= E_L^j \sin\theta_j + E_\parallel^j \cos\theta_j \\ &= -\mathfrak{n}_j E_y^j \cos\theta_j & &= -Q_j E_y^j \sin\theta_j + \frac{1}{\rho_j} E_y^j \cos\theta_j \end{aligned} \tag{97}$$

$$\begin{aligned} \mathscr{H}_y^j &= \mathfrak{n}_j E_\parallel^j & E_y^j &= E_y^j \\ &= \frac{\mathfrak{n}_j}{\rho_j} E_y^j \end{aligned} \tag{98}$$

$$\begin{aligned} \mathscr{H}_z^j &= \mathscr{H}_\parallel^j \sin\theta_j & E_z^j &= E_L^j \cos\theta_j - E_\parallel^j \sin\theta_j \\ &= \mathfrak{n}_j E_y^j \sin\theta_j & &= -Q_j E_y^j \cos\theta_j - \frac{1}{\rho_j} E_y^j \sin\theta_j \end{aligned} \tag{99}$$

5.2.4 continuity of tangential fields across boundary

At this point in the calculation, the total tangential fields just below and just above the vacuum/plasma boundary must be matched. That is, the x and y field components must be equated at the $z = 0$ plane:

$$\begin{aligned} E_x &\longrightarrow & E_x^i + E_x^r &= E_x^a + E_x^b \\ E_y &\longrightarrow & E_y^i + E_y^r &= E_y^a + E_y^b \\ \mathscr{H}_x &\longrightarrow & \mathscr{H}_x^i + \mathscr{H}_x^r &= \mathscr{H}_x^a + \mathscr{H}_x^b \\ \mathscr{H}_y &\longrightarrow & \mathscr{H}_y^i + \mathscr{H}_y^r &= \mathscr{H}_y^a + \mathscr{H}_y^b \end{aligned} \tag{100}$$

Using equations 88 - 90, 91 - 93, and 97 - 99, results in

$$\left(E_\parallel^i - E_\parallel^r\right)\cos\theta_i = \frac{E_y^a}{\rho_a}\cos\theta_a - Q_a E_y^a \sin\theta_a + \frac{E_y^b}{\rho_b}\cos\theta_b - Q_b E_y^b \sin\theta_b \tag{101}$$

$$E_y^i + E_y^r = E_y^a + E_y^b \tag{102}$$

$$\left(E_y^r - E_y^i\right)\cos\theta_i = -q_a E_y^a - q_b E_y^b \tag{103}$$

$$E_\parallel^i + E_\parallel^r = \frac{\mathfrak{n}_a}{\rho_a} E_y^a + \frac{\mathfrak{n}_b}{\rho_b} E_y^b \tag{104}$$

where the definition of q from eqn. 67 was used in eqn. 103.

This is a system of four equations with four unknowns - $E_\parallel^r, E_y^r, E_y^a$, and E_y^b. All other field quantities are found from the local plasma conditions, Booker quartic, and the polarization relations (94).

The method for solving these equations will be put off until the next section, where the magnetic meridian plane will not be constrained to lie along principal axes.

5.3 case II: magnetic meridian plane oblique (general case)

Section 5.2 outlined the calculation for propagating the incident EM wave fields across the vacuum/plasma boundary. However, the magnetic meridian plane and vacuum/plasma interface plane were constrained to be along principal axes of the coordinate system. This was done to illustrate the process of finding the transmitted and reflected wave fields without over complicating the math. With that knowledge, the more general case of a completely oblique magnetic field vector relative to the incident wave can be solved.

5.3.1 orientation at the interface

The issue is basically coordinate system rotation. Figure 7 shows the general case for oblique incidence and an obliquely oriented magnetic field vector. In section 5.2 it was assumed that the incident wave normal and magnetic field vectors were co-planar along principal axes, with the vacuum/plasma boundary forming the resulting right-handed coordinate system. In this case, that assumption is dropped and a different coordinate rotation will be used to facilitate calculation of the transmitted and reflected fields.

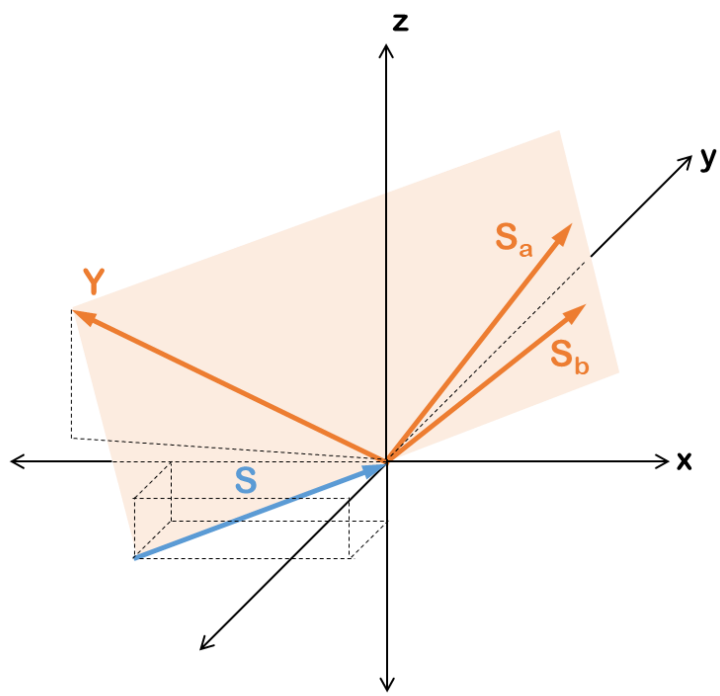

Figure 7: *The orientation for oblique incident and refracted waves in a collisionless, magnetized plasma.* $\mathbf{S}$ *is incident from below the vacuum/plasma interface ($x-y$ plane). It will refract into two modes* $\mathbf{S_a}$ *and* $\mathbf{S_b}$ *(assuming neither is cutoff) in the* $\mathbf{S}-\mathbf{Y}$ *plane.*

Orientation at the point of incidence can always be rotated into the geometry of Fig. 8(a). Here the vector $\mathbf{Y}$ is oblique, and the refracted part of $\mathbf{S}$ lies in a principal axis plane perpendicular to the vacuum/plasma interface plane. This will be labeled the unprimed coordinate system. In section 5.2, this system was rotated until the magnetic field and wave normal vectors were coplanar along principal axes, resulting in the orientation of figure 6. While this resulted in a more straightforward way to illustrate the problem, a different coordinate rotation will result in a 'cleaner' problem to be solved.

This is shown in figure 8(b). Here, the new primed coordinate system has **S** along the **z** axis, normal to the vacuum/plasma interface plane, with **Y** in the $\mathbf{x}-\mathbf{z}$ plane. Solutions for the reflected and transmitted fields can then be rotated into the original geometry. Note that the incident, transmitted, and reflected angles will all be zero in this coordinate system. Also, the coordinate rotation will be necessary for each refracted mode, since they will refract at different angles.

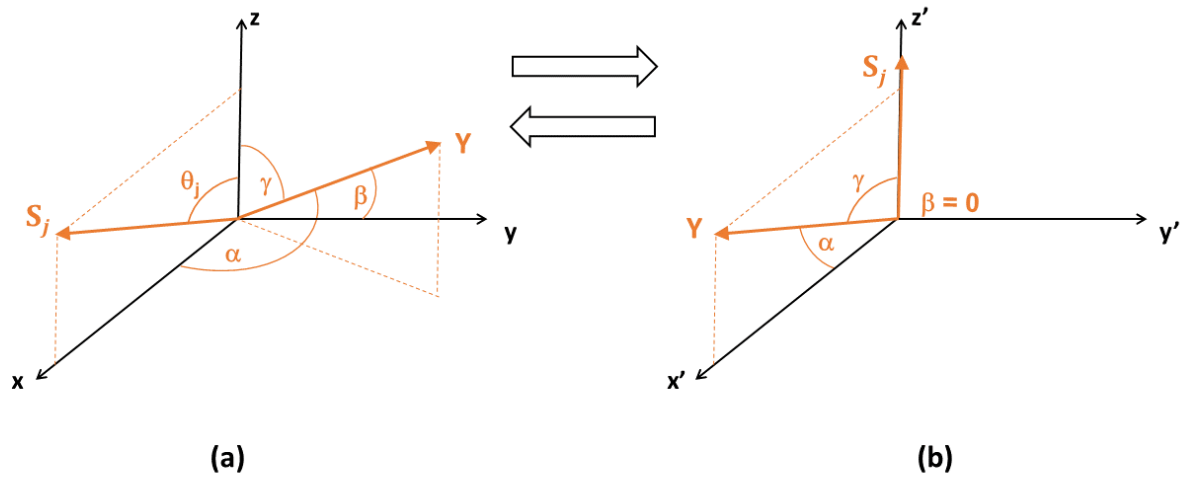

Figure 8: *Just inside the plasma. Coordinate rotation from oblique unprimed system to final primed system. The unprimed direction cosines* (l, m, n) *are* $(\cos\alpha, \cos\beta, \cos\gamma)$ *and the refracted angle* θ_j *is known. In the primed coordinate system,* (l, m, n) *are* $(\cos\alpha, 0, \cos\beta)$*, and* $\theta_j = 0$*. The direction cosines for* **S** *in the unprimed coordinate system are* $(l, m, n) = (\sin\theta_j, 0, \cos\theta_j.)$

5.3.2 coordinate rotation matrix

To get from the system of Fig. 8(a) to Fig. 8(b), the 3×3 coordinate rotation matrix must be found. Recall that two transformation matrices must be found, one for each mode. For this part, assume a single mode and drop any associated subscripts. All vector quantities will be associated with locations inside the plasma, thus **S** will refer to the refracted part of the incident wave.

Let $\hat{a}_3$ be a unit vector in the direction of **S** in the unprimed coordinate system.

$$\hat{a}_3 = 1 \cdot (\sin\theta, 0, \cos\theta) \tag{105}$$

Now form a unit vector $\hat{a}_2$ perpendicular to the $\mathbf{S}-\mathbf{Y}$ plane. Start by forming the vector $\overrightarrow{a_2}$ from the cross product of $\hat{a}_3$ and **Y**.

$$\begin{aligned}\overrightarrow{a_2} &= \hat{a}_3 \times \mathbf{Y} \\ &= Y\,(-m\cos\theta,\;\; l\cos\theta - n\sin\theta,\;\; m\sin\theta)\end{aligned} \tag{106}$$

Its magnitude is

$$\begin{aligned}\left|\overrightarrow{a_2}\right| &= Y\left(m^2\cos^2\theta + l^2\cos^2\theta + n^2\sin^2\theta - 2ln\cos\theta\sin\theta + m^2\sin^2\theta\right)^{1/2} \\ &= Y\left[1-(l\sin\theta + n\cos\theta)^2\right]^{1/2} \\ &= YG \end{aligned} \tag{107}$$

where the identity for direction cosines, $l^2 + m^2 + n^2 = 1$, was used. The unit vector $\hat{a}_2$ is therefore

$$\begin{aligned}\hat{a}_2 &= \frac{\overrightarrow{a_2}}{\left|\overrightarrow{a_2}\right|} \\ &= \left(-\frac{m\cos\theta}{G},\ \frac{l\cos\theta - n\sin\theta}{G},\ \frac{m\sin\theta}{G}\right)\end{aligned} \tag{108}$$

To complete the right handed primed coordinate system, the unit vector $\hat{a}_1$ is found from the cross product of $\hat{a}_2$ and $\hat{a}_3$

$$\begin{aligned}\hat{a}_1 &= \hat{a}_2 \times \hat{a}_3 \\ &= \frac{l\cos\theta - n\sin\theta}{G}\cos\theta,\ \frac{m}{G},\ \frac{l\cos\theta - n\sin\theta}{G}(-\sin\theta)\end{aligned} \tag{109}$$

The primed coordinate system has unit vectors $\hat{a}_1,\ \hat{a}_2,\ \hat{a}_3$ corresponding to $\hat{x}',\ \hat{y}',\ \hat{z}'$ (see figure 8), from which the rotation matrix, $\mathbb{M}$, to transform from one system to the other can be constructed [13]. Thus, for some vector $\overrightarrow{r}$ in the original coordinate system

$$\overrightarrow{r}' = \mathbb{M}\cdot\overrightarrow{r} = \begin{bmatrix}\hat{a_{1x}} & \hat{a_{1y}} & \hat{a_{1z}} \\ \hat{a_{2x}} & \hat{a_{2y}} & \hat{a_{2z}} \\ \hat{a_{3x}} & \hat{a_{3y}} & \hat{a_{3z}}\end{bmatrix}\cdot\overrightarrow{r} \tag{110}$$

and, likewise

$$\overrightarrow{r} = \mathbb{M}^{-1}\cdot\overrightarrow{r}' \tag{111}$$

where the matrix $\mathbb{M}$ is

$$\mathbb{M} = \begin{bmatrix}\frac{l\cos\theta - n\sin\theta}{G}\cos\theta & \frac{m}{G} & \frac{l\cos\theta - n\sin\theta}{G}(-\sin\theta) \\ -\frac{m\cos\theta}{G} & \frac{l\cos\theta - n\sin\theta}{G} & \frac{m\sin\theta}{G} \\ \sin\theta & 0 & \cos\theta\end{bmatrix} \tag{112}$$

The quantites G and θ are mode specific, and thus a rotation matrix for each mode must be calculated in order to transform the fields of each mode from the primed to unprimed coordinate systems (and vice versa).

$$\begin{aligned}\theta_a, G_a &\longrightarrow \mathbb{M}_a \\ \theta_b, G_b &\longrightarrow \mathbb{M}_b\end{aligned}$$

Table 1 lists the quantities that are *invariant* to coordinate rotations. They depend on local plasma density and magnetic field values only. This is very helpful when constructing the rotation matrix for each mode.

Quantity	Equation(s)
q_j	75
$\mathfrak{n}_j$	68
θ_j	68
Y_L^j	72, 95
Y_T^j	96
ρ_j	94
Q_j	94

Table 1: *Quantities that are invariant to coordinate rotation.*

The solution for the transmitted fields in this case can now be found as before in section 5.2.3, but with an extra step. Using figure 9 for reference, write down the form of the transmitted fields for each mode in the primed coordinate system. Note that in this coordinate system, the transmitted wave normal angle for each mode zero; also, the subscript $\|$ is assigned to the x direction. Next, rotate them into the unprimed coordinate system via equation 111 - this is the new step. The tangential components of these rotated fields are equated to the tangential components of the incident and reflected fields using Snell's law. This will generate a new set of equations that are solved to give the final form of the transmitted fields for each mode.

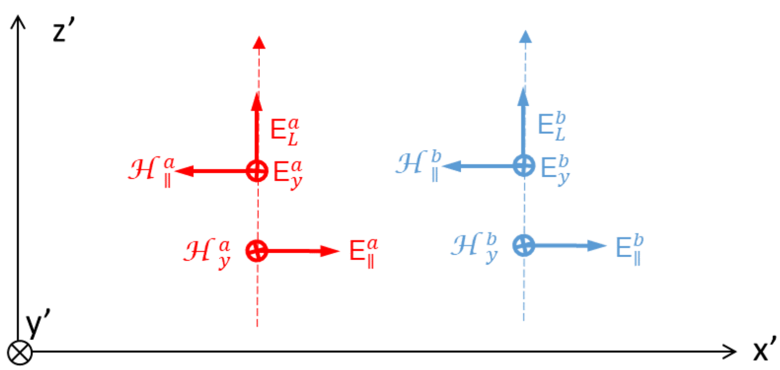

Figure 9: *Transmitted fields just above the vacuum/plasma interface - primed coordinate system.*

5.3.3 incident and reflected fields

The incident and reflected wave fields for this case are the same as those found in section 5.2.1 equations 88, 89, 90; and section 5.2.2, equations 91, 92, 93.

5.3.4 transmitted fields

The transmitted fields in the primed coordinate system are (dropping the ’ notation for convenience)

$$\begin{aligned} \mathscr{H}_x^j &= -\mathscr{H}_\parallel^j & \qquad E_x^j &= \frac{1}{\rho_j} E_y^j \\ &= -\mathfrak{n}_j E_y^j \end{aligned} \tag{113}$$

$$\begin{aligned} \mathscr{H}_y^j &= \mathfrak{n}_j E_\parallel^j & \qquad E_y^j &= E_y^j \\ &= \frac{\mathfrak{n}_j}{\rho_j} E_y^j \end{aligned} \tag{114}$$

$$\begin{aligned} \mathscr{H}_z^j &= 0 & \qquad E_z^j &= E_L^j \\ & & &= -Q_j E_y^j \end{aligned} \tag{115}$$

5.3.5 continuity of tangential fields at the interface

The first order of business is to rotate the transmitted fields to the unprimed coordinate system. Let $\mathbb{m}$ represent the inverse of the matrix $\mathbb{M}$

$$\mathbb{m} = \mathbb{M}^{-1} \tag{116}$$

Then in the plasma

$$\mathbf{E}^j = \mathbb{m} \cdot \mathbf{E}'^j \tag{117}$$

$$\overrightarrow{\mathscr{H}}^j = \mathbb{m} \cdot \overrightarrow{\mathscr{H}'}^j \tag{118}$$

and the equations for the plasma wave fields in the unprimed coordinate system, for each mode, are (dropping the j superscript)

$$\begin{bmatrix} \mathscr{H}_x \\ \mathscr{H}_y \\ \mathscr{H}_z \end{bmatrix} = \begin{bmatrix} \mathbb{m}_{11} & \mathbb{m}_{12} & \mathbb{m}_{13} \\ \mathbb{m}_{21} & \mathbb{m}_{22} & \mathbb{m}_{23} \\ \mathbb{m}_{31} & \mathbb{m}_{32} & \mathbb{m}_{33} \end{bmatrix} \cdot \begin{bmatrix} \mathscr{H}'_x \\ \mathscr{H}'_y \\ \mathscr{H}'_z \end{bmatrix} \tag{119}$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \mathbb{m}_{11} & \mathbb{m}_{12} & \mathbb{m}_{13} \\ \mathbb{m}_{21} & \mathbb{m}_{22} & \mathbb{m}_{23} \\ \mathbb{m}_{31} & \mathbb{m}_{32} & \mathbb{m}_{33} \end{bmatrix} \cdot \begin{bmatrix} E'_x \\ E'_y \\ E'_z \end{bmatrix} \tag{120}$$

This gives a system of six equations in six unknowns. The number of unknowns can be reduced significantly using the polarization relations for the E-fields (equations 94) and Faraday’s law (equation 87) for PPW’s traveling in an anisotropic homogeneous medium. Furthermore, the parameters used in those equations are invariant to coordinate system rotation (see Table 1).

$$E'^j_x = \frac{1}{\rho_j} E'^j_y \qquad\qquad E'^j_z = -Q_j E'^j_y \tag{121}$$

$$\mathscr{H}'^j_x = -\mathfrak{n}_j E'^j_y \qquad \mathscr{H}'^j_y = \frac{\mathfrak{n}_j}{\rho_j} E'^j_y \qquad \mathscr{H}'^j_z = 0 \tag{122}$$

Using the above relations and the equations in Table 1, the wave fields in the plasma for each mode can be found in the unprimed coordinate system as functions of the unknown field quantity $E_y^{\prime j}$.

$$E_x^j = \left[\mathbb{m}_{11}^j \left(\frac{1}{\rho_j} \right) + \mathbb{m}_{12}^j - \mathbb{m}_{13}^j Q_j \right] E_y^{\prime j} \tag{123}$$

$$E_y^j = \left[\mathbb{m}_{21}^j \left(\frac{1}{\rho_j} \right) + \mathbb{m}_{22}^j - \mathbb{m}_{23}^j Q_j \right] E_y^{\prime j} \tag{124}$$

$$E_z^j = \left[\mathbb{m}_{31}^j \left(\frac{1}{\rho_j} \right) + \mathbb{m}_{32}^j - \mathbb{m}_{33}^j Q_j \right] E_y^{\prime j} \tag{125}$$

$$\mathcal{H}_x^j = \left[-\mathbb{m}_{11}^j \mathfrak{n}_j + \mathbb{m}_{12}^j \left(\frac{\mathfrak{n}_j}{\rho_j} \right) \right] E_y^{\prime j} \tag{126}$$

$$\mathcal{H}_y^j = \left[-\mathbb{m}_{21}^j \mathfrak{n}_j + \mathbb{m}_{22}^j \left(\frac{\mathfrak{n}_j}{\rho_j} \right) \right] E_y^{\prime j} \tag{127}$$

$$\mathcal{H}_z^j = \left[-\mathbb{m}_{31}^j \mathfrak{n}_j + \mathbb{m}_{32}^j \left(\frac{\mathfrak{n}_j}{\rho_j} \right) \right] E_y^{\prime j} \tag{128}$$

Now the tangential incident, reflected, and transmitted fields in the unprimed coordinate system for this case can be matched at the vacuum/plasma boundary via Snell's law (see equations 100). Specifically, the incident tangential fields given in equations 88 and 89; the reflected tangential fields from equations 91 and 92; and the transmitted fields for each mode from equations 123, 124, 126, and 127. For completeness, they are listed below

incident fields

$$\mathcal{H}_x^i = -E_y^i \cos\theta_i \qquad E_x^i = E_\parallel^i \cos\theta_i \tag{129}$$

$$\mathcal{H}_y^i = E_\parallel^i \qquad E_y^i = E_y^i \tag{130}$$

$$\mathcal{H}_z^i = -E_y^i \sin\theta_i \qquad E_z^i = -E_\parallel^i \sin\theta_i \tag{131}$$

reflected fields

$$\mathcal{H}_x^r = E_y^r \cos\theta_i \qquad E_x^r = -E_\parallel^r \cos\theta_i \tag{132}$$

$$\mathcal{H}_y^r = E_\parallel^r \qquad E_y^r = E_y^r \tag{133}$$

$$\mathcal{H}_z^r = -E_y^r \sin\theta_i \qquad E_z^r = -E_\parallel^r \sin\theta_i \tag{134}$$

transmitted fields

$$E_x^a = \left[\mathfrak{m}_{11}^a \frac{1}{\rho_a} + \mathfrak{m}_{12}^a - \mathfrak{m}_{13}^a Q_a\right] E'^a_y \qquad = A_1 E'^a_y \tag{135}$$

$$E_x^b = \left[\mathfrak{m}_{11}^b \frac{1}{\rho_b} + \mathfrak{m}_{12}^b - \mathfrak{m}_{13}^b Q_b\right] E'^b_y \qquad = B_1 E'^b_y \tag{136}$$

$$E_y^a = \left[\mathfrak{m}_{21}^a \frac{1}{\rho_a} + \mathfrak{m}_{22}^a - \mathfrak{m}_{23}^a Q_a\right] E'^a_y \qquad = A_2 E'^a_y \tag{137}$$

$$E_y^b = \left[\mathfrak{m}_{21}^b \frac{1}{\rho_b} + \mathfrak{m}_{22}^b - \mathfrak{m}_{23}^b Q_b\right] E'^b_y \qquad = B_2 E'^b_y \tag{138}$$

$$E_z^a = \left[\mathfrak{m}_{31}^a \frac{1}{\rho_a} + \mathfrak{m}_{32}^a - \mathfrak{m}_{33}^a Q_a\right] E'^a_y \qquad = A_3 E'^a_y \tag{139}$$

$$E_z^b = \left[\mathfrak{m}_{31}^b \frac{1}{\rho_b} + \mathfrak{m}_{32}^b - \mathfrak{m}_{33}^b Q_b\right] E'^b_y \qquad = B_3 E'^b_y \tag{140}$$

$$\mathscr{H}_x^a = \left[-\mathfrak{n}_a \mathfrak{m}_{11}^a + \frac{\mathfrak{n}_a}{\rho_a}\mathfrak{m}_{12}^a\right] E'^a_y \qquad = A_4 E'^a_y \tag{141}$$

$$\mathscr{H}_x^b = \left[-\mathfrak{n}_b \mathfrak{m}_{11}^b + \frac{\mathfrak{n}_b}{\rho_b}\mathfrak{m}_{12}^b\right] E'^b_y \qquad = B_4 E'^b_y \tag{142}$$

$$\mathscr{H}_y^a = \left[-\mathfrak{n}_a \mathfrak{m}_{21}^a + \frac{\mathfrak{n}_a}{\rho_a}\mathfrak{m}_{22}^a\right] E'^a_y \qquad = A_5 E'^a_y \tag{143}$$

$$\mathscr{H}_y^b = \left[-\mathfrak{n}_b \mathfrak{m}_{21}^b + \frac{\mathfrak{n}_b}{\rho_b}\mathfrak{m}_{22}^b\right] E'^b_y \qquad = B_5 E'^b_y \tag{144}$$

$$\mathscr{H}_z^a = \left[-\mathfrak{n}_a \mathfrak{m}_{31}^a + \frac{\mathfrak{n}_a}{\rho_a}\mathfrak{m}_{32}^a\right] E'^a_y \qquad = A_6 E'^a_y \tag{145}$$

$$\mathscr{H}_z^b = \left[-\mathfrak{n}_b \mathfrak{m}_{31}^b + \frac{\mathfrak{n}_b}{\rho_b}\mathfrak{m}_{32}^b\right] E'^b_y \qquad = B_6 E'^b_y \tag{146}$$

To proceed, let $C = \cos\theta_i$ and $S = \sin\theta_i$, and equate the x and y components of the E and $\mathscr{H}$ fields. This results in the following system of four equations in four unknowns

$$\begin{bmatrix} C & 0 & A_1 & B_1 \\ 0 & -1 & A_2 & B_2 \\ 0 & -C & A_4 & B_4 \\ -1 & 0 & A_5 & B_5 \end{bmatrix} \cdot \begin{bmatrix} E^r_\parallel \\ E^r_y \\ E'^a_y \\ E'^b_y \end{bmatrix} = \begin{bmatrix} CE^i_\parallel \\ E^i_y \\ -CE^i_y \\ E^i_\parallel \end{bmatrix} \tag{147}$$

This system is solved using Cramer's rule [14], giving the reflected wave components, and the y component of the two refracted modes in the primed coordinate system.

The results are summarized below. The determinant of the 4×4 coefficient matrix in Eqn. 147 is

$$\Delta = C^2 (A_2 B_5 - B_2 A_5) + C (A_5 B_4 - A_4 B_5 + B_1 A_2 - A_1 B_2) + A_1 B_4 - B_1 A_4 \tag{148}$$

To solve for the first variable in the 4×1 column matrix on the left hand side of equation 147 ($E^r_\parallel$), substitute the right hand side 4×1 column matrix into the first column of the 4×4 coefficient matrix. This gives a resultant matrix from which the determinant must be found

$$\Delta_1 = \begin{vmatrix} CE^i_\parallel & 0 & A_1 & B_1 \\ E^i_y & -1 & A_2 & B_2 \\ -CE^i_y & -C & A_4 & B_4 \\ E^i_\parallel & 0 & A_5 & B_5 \end{vmatrix} \tag{149}$$

It is most convenient expand along the second column to get

$$\Delta_1 = \Big[C^2 E^i_\parallel (A_2 B_5 - A_5 B_2) + CE^i_\parallel (A_1 B_2 - B_1 A_2 + A_5 B_4 - A_4 B_5) \tag{150}$$

$$+2CE^i_y (B_1 A_5 - A_1 B_5) + E^i_\parallel (B_1 A_4 - A_1 B_4) \Big] \tag{151}$$

Then $E^r_\parallel$ is [14]

$$\begin{aligned} E^r_\parallel &= \frac{\Delta_1}{\Delta} \\ &= \frac{1}{\Delta} \Big[C^2 E^i_\parallel (A_2 B_5 - A_5 B_2) + CE^i_\parallel (A_1 B_2 - B_1 A_2 + A_5 B_4 - A_4 B_5) \\ &\quad +2CE^i_y (B_1 A_5 - A_1 B_5) + E^i_\parallel (B_1 A_4 - A_1 B_4) \Big] \end{aligned} \tag{152}$$

For E^r_y, it is more convenient to expand along the first column of the resultant coefficient matrix

$$\Delta_2 = \begin{vmatrix} C & CE^i_\parallel & A_1 & B_1 \\ 0 & E^i_y & A_2 & B_2 \\ 0 & -CE^i_y & A_4 & B_4 \\ -1 & E^i_\parallel & A_5 & B_5 \end{vmatrix} \tag{153}$$

$$\begin{aligned} E^r_y &= \frac{\Delta_2}{\Delta} \\ &= \frac{1}{\Delta} \Big[C^2 E^i_y (A_2 B_5 - A_5 B_2) + CE^i_y (A_4 B_5 - B_4 A_5 + A_2 B_1 - A_1 B_2) \\ &\quad +2CE^i_\parallel (B_4 A_2 - A_4 B_2) + E^i_y (B_1 A_4 - A_1 B_4) \Big] \end{aligned} \tag{154}$$

E'^a_y and E'^b_y are also more easily calculated by using the determinants found from expanding along the first column of their respective resultant matrices

$$\Delta_3 = \begin{vmatrix} C & 0 & CE^i_\parallel & B_1 \\ 0 & -1 & E^i_y & B_2 \\ 0 & -C & -CE^i_y & B_4 \\ -1 & 0 & E^i_\parallel & B_5 \end{vmatrix} \qquad \Delta_4 = \begin{vmatrix} C & 0 & A_1 & CE^i_\parallel \\ 0 & -1 & A_2 & E^i_y \\ 0 & -C & A_4 & -CE^i_y \\ -1 & 0 & A_5 & E^i_\parallel \end{vmatrix} \tag{155}$$

$$
\begin{aligned}
E'^{a}_{y} &= \frac{\Delta_3}{\Delta} \\
&= \frac{2}{\Delta}\left[C^2\left(E^i_y B_5 - E^i_\parallel B_2\right) + C\left(E^i_\parallel B_4 + E^i_y B_1\right)\right]
\end{aligned} \tag{156}
$$

$$
\begin{aligned}
E'^{b}_{y} &= \frac{\Delta_4}{\Delta} \\
&= \frac{2}{\Delta}\left[C^2\left(E^i_\parallel A_2 - E^i_y A_5\right) - C\left(E^i_\parallel A_4 - E^i_y A_1\right)\right]
\end{aligned} \tag{157}
$$

The polarization relations, equations 94, will give the other wave field spatial components for each mode in the primed coordinate system. For the wave field components in the unprimed system, the E-fields are given in equations 135 - 140, and the $\mathscr{H}$-fields are given in equations 141 - 146.

5.3.6 conservation of energy flux perpendicular to the boundary

Inherent loss mechanisms in this system have been assumed to be negligible. Basically, electron collisions have been neglected. This is a very good assumption for the frequencies pertinent to the situations addressed by the calculations in this report. This also allows a method to check the validity of the refracted wave field calculations.

The energy flux in the wave incident on a boundary between two lossless media, perpendicular to the interface boundary, is conserved [4, 5]. In other words, the total Poynting vectors in the direction perpendicular to the interface boundary should be equal in magnitude (incident power minus the reflected power should equal the transmitted power) at the boundary.

The Poynting vector, in the notation adopted for this report, is [4]

$$
\mathbf{P} = \frac{1}{2\eta_0}\mathrm{Re}\left\{\mathbf{E} \times \overrightarrow{\mathscr{H}^*}\right\} \tag{158}
$$

The components of this vector perpendicular to the interface plane in the unprimed coordinate system are then

$$
P_z = \frac{1}{2\eta_0}\mathrm{Re}\left\{E_x \mathscr{H}^*_y - E_y \mathscr{H}^*_x\right\} \tag{159}
$$

Using this, the incident, reflected, and transmitted (both modes) components can be calculated and checked for correctness.

$$
P^i_z - P^r_z = P^a_z + P^b_z \tag{160}
$$

5.4 special cases

In section 5.3, solutions for the refracted wave fields for each birefringent mode in the magnetized ionospheric plasma are found, along with the reflected fields. These solutions were found by rotating into an advantageous coordinate system, allowing the problem to be recast such that a straightforward, and easier, path to the solution was demonstrated. There are certain special cases where no coordinate rotation is necessary, or where the rotation matrix $\mathbb{M}$ would include some infinite members of the form $1/0$. In these cases, the problem must be cast differently.

5.4.1 case 1: incident wave perpendicular and magnetic field horizontal at interface plane

In this situation, the incident wave vector **S** lies along the z-axis, and the magnetic field vector **Y** lies along the x-axis, as shown in figure 10.

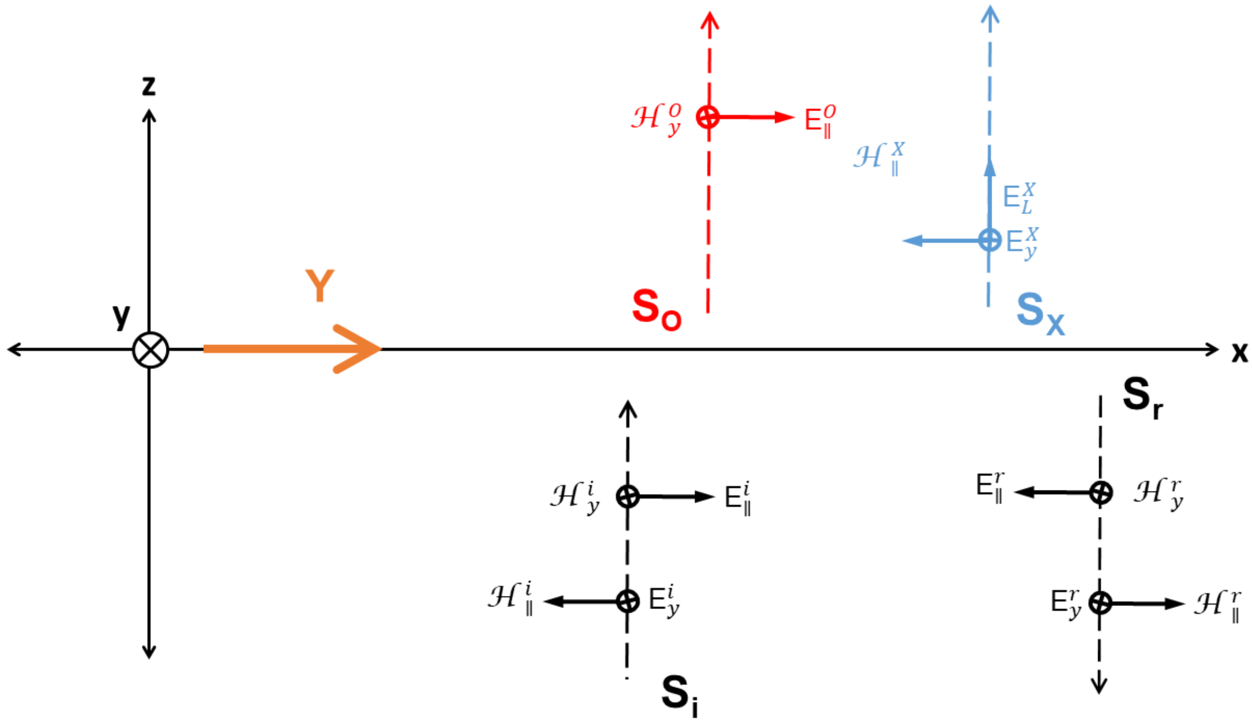

Figure 10: *Perpendicular wave incidence along* $\hat{z}$ *with* $\theta_i = \theta_r = \theta_{O,X} = 0$. $\mathbf{Y} = Y \cdot (1, 0, 0)$ *and* $\mathbf{S} = S \cdot (0, 0, 1)$. *On the plasma side: Red = ordinary mode, Blue = extraordinary mode.*

The incident, reflected, and transmitted angles are all zero. The term G in equation 107 becomes zero, and many terms in the coordinate rotation matrix in equation 112 become infinite. Thus, the problem must be recast in order to solve for the transmitted and reflected wave fields. Indeed, the situation as illustrated in Fig. 10 shows that there is no need for a coordinate rotation.

Notice also that this situation involves wave propagation in the plasma exactly perpendicular to the magnetic field. Here the two modes that can propagate in the plasma are identified as two principal modes since the wave normal vector **S** and magnetic field vector **Y** are exactly perpendicular [2, 1] (the other two principal modes identified when **S** and **Y** are exactly parallel or anti-parallel).

The two propagating modes are identified by the orientation of their E-field with respect to the magnetic field vector **Y**. The first mode (the fast wave root of the CPDR, equation 57), called the *ordinary* mode, has its E-field aligned parallel (or anti-parallel) to the magnetic field vector **Y**. This is also the same mode that propagates in an unmagnetized plasma [1, 2, 3]. This particular mode does not have an electrostatic component, that is, $E_L \to 0$. The second mode (the slow wave root of the CPDR, equation 57) is identified as the *extrordinary* mode, and has its E-field aligned perpendicular to **Y**. This mode has an electrostatic component along the wave normal direction. These two waves are illustrated in Fig. 10.

The use of reflection and transmission coefficients at the vacuum/plasma interface was not necessary in section 5.3 because both reflected and transmitted fields were included in the Cramer's method solution. Any spatial wave field component in the incident wave can contribute to *both* modes on the plasma side, and the Cramer's method solution is more straightforward than implementing the necessary reflection and transmission coefficients [4]. For reference, there are eight required coefficients, given in Table 2.

incident field component	contributes to	coefficient
$\parallel$	reflected $\parallel$	${}_{\parallel}R_{\parallel}$
$\perp$	reflected $\perp$	${}_{\perp}R_{\perp}$
$\parallel$	reflected $\perp$	${}_{\parallel}R_{\perp}$
$\perp$	reflected $\parallel$	${}_{\perp}R_{\parallel}$
$\parallel$	transmitted mode a	${}_{\parallel}T_a$
$\parallel$	transmitted mode b	${}_{\parallel}T_b$
$\perp$	transmitted mode a	${}_{\perp}T_a$
$\perp$	transmitted mode b	${}_{\perp}T_b$

Table 2: *The 8 field coefficients required to solve for the reflected and transmitted fields in section 5.3 if Cramer's method is not employed.*

However, in this case there is no cross-coupling in reflection or transmission. One spatial component of the incident wave field will only contribute to one mode on the plasma, not both; and reflection in one polarization will not contribute to a different polarization. For example $E^i_{\parallel}$ can only drive the ordinary mode, E^i_y can only drive the extraordinary mode, and they will only reflect into their respective polarizations. It is straightforward to show that, for this case,

$$\rho_a \to \rho_O = 0 \qquad \rho_b \to \rho_X = \infty \tag{161}$$

$$Q_a \to Q_O = 0 \qquad Q_b \to Q_X \neq 0 \tag{162}$$

$$\mathfrak{n}_a \to \mathfrak{n}_O \qquad \mathfrak{n}_b \to \mathfrak{n}_X \tag{163}$$

Furthermore, there are only two nonzero reflection coefficients [4]

$${}_{\parallel}R_{\perp} = 0 \qquad {}_{\perp}R_{\parallel} = 0 \tag{164}$$

$${}_{\parallel}R_{\parallel} = \frac{\mathfrak{n}_a - 1}{\mathfrak{n}_a + 1} \qquad {}_{\perp}R_{\perp} = \frac{1 - \mathfrak{n}_b}{1 + \mathfrak{n}_b} \tag{165}$$

With these definitions the incident, reflected and transmitted wave fields can be listed, referring to figure 10, and using Faraday's law from equation 87.

incident fields

$$\mathscr{H}^i_x = -E^i_y \qquad E^i_x = E^i_{\parallel} \tag{166}$$

$$\mathscr{H}^i_y = E^i_{\parallel} \qquad E^i_y = E^i_y \tag{167}$$

$$\mathscr{H}^i_z = 0 \qquad E^i_z = 0 \tag{168}$$

reflected fields

$$\mathcal{H}_x^r = \left(\frac{1-\mathfrak{n}_b}{1+\mathfrak{n}_b}\right) E_y^i \qquad E_x^r = -\left(\frac{\mathfrak{n}_a-1}{\mathfrak{n}_a+1}\right) E_\parallel^i \tag{169}$$

$$\mathcal{H}_y^r = \left(\frac{\mathfrak{n}_a-1}{\mathfrak{n}_a+1}\right) E_\parallel^i \qquad E_y^r = \left(\frac{1-\mathfrak{n}_b}{1+\mathfrak{n}_b}\right) E_y^i \tag{170}$$

$$\mathcal{H}_z^r = 0 \qquad E_z^r = 0 \tag{171}$$

transmitted fields

$$\mathcal{H}_x = \mathcal{H}_x^b = -\mathfrak{n}_b E_y^b \qquad E_x = E_x^a \tag{172}$$

$$\mathcal{H}_y = \mathcal{H}_y^a = \mathfrak{n}_a E_x^a \qquad E_y = E_y^b \tag{173}$$

$$\mathcal{H}_z = 0 \qquad E_z = E_z^b = Q_b E_y^b \tag{174}$$

continuity of tangential fields at the interface The sum of the x and y components of the incident (equations 166, 167) and reflected (equations 169, 170) wave fields is equated to the respective transmitted wave fields (equations 172, 173) resulting in an over determined system of equations with four equations and two unknowns.

$$E_x \longrightarrow \left(1 -_\parallel R_\parallel\right) E_x^i = E_x^a \tag{175}$$

$$E_y \longrightarrow \left(1 +_\perp R_\perp\right) E_y^i = E_y^b \tag{176}$$

$$\mathcal{H}_x \longrightarrow \left({}_\perp R_\perp - 1\right) E_y^i = -\mathfrak{n}_b E_y^b \tag{177}$$

$$\mathcal{H}_y \longrightarrow \left(1 +_\parallel R_\parallel\right) E_x^i = \mathfrak{n}_a E_x^a \tag{178}$$

From these, take one x, and one y component equation and solve for the two unknown wave fields E_x^a and E_y^b

$$E_x^a = \frac{2}{\mathfrak{n}_a+1} E_x^i \tag{179}$$

$$E_y^b = \frac{2}{\mathfrak{n}_b+1} E_y^i \tag{180}$$

$$\tag{181}$$

Recall, from the right side of equation 174 that the longitudinal component of the transmitted wave field is

$$E_z = E_z^b = Q_b E_y^b = \frac{2Q_b}{\mathfrak{n}_b+1} E_y^i = -\frac{i2Y(\mathfrak{n}_b-1)}{1-X} E_y^i \tag{182}$$

The solutions for $\mathcal{H}_x^b$ and $\mathcal{H}_y^a$ can be found from the left side of equations 172 and 173.

$$\mathcal{H}_y^a = \frac{2\mathfrak{n}_a}{\mathfrak{n}_a+1} E_x^i \tag{183}$$

$$\mathcal{H}_x^b = \frac{-2\mathfrak{n}_b}{\mathfrak{n}_b+1} E_y^i \tag{184}$$

Now all the transmitted wave fields are specified for this case.

5.4.2 case 2: incident wave oblique and magnetic field horizontal at interface plane, perpendicular to incidence plane

In this situation the incident wave is oblique in a plane perpendicular to the magnetic field vector, which lies in the interface plane. This is shown in Fig. 11. This is basically the same situation as the first special case, where the plane of incidence is the $\mathbf{S}-\mathbf{Y}$ plane, except that two coordinate rotations separate this geometry from that in the first special case. It is evident that the incident wave is again comprised of pure X and O modes, and will refract into pure X and O modes; where as in special case 1, the mode designations are $a \to O$ and $b \to X$.

Solutions for the transmitted and reflected wave fields can be found by rotating this system into the system from special case 1 and using those field solutions. Here, the primed coordinate system will be the coordinate system from special case 1.

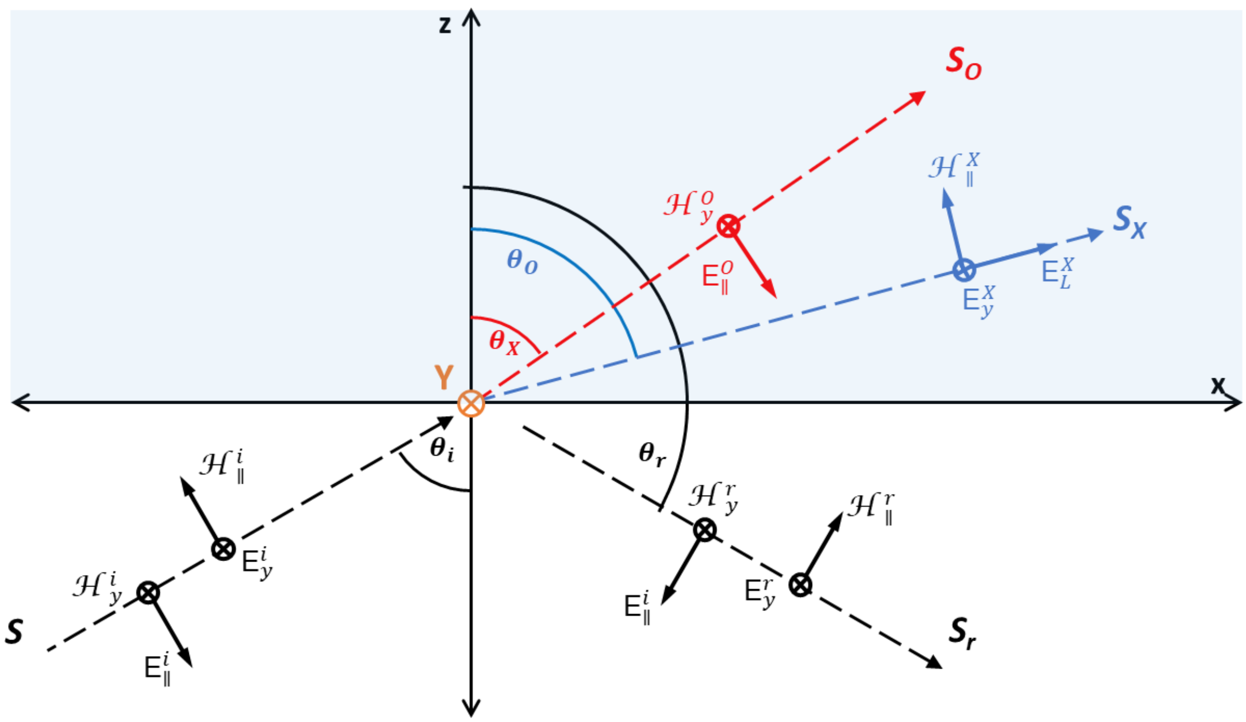

Figure 11: *Oblique wave incidence in the $x-z$ plane.* $\mathbf{Y} = Y \cdot (0,1,0)$ *and* $\mathbf{S} = S \cdot (\sin\theta_i, 0, \cos\theta_i)$.

As before, let $C = \cos\theta_i$ and $S = \sin\theta_i$. Transforming from the unprimed to the primed system will require a rotation about the y-axis by θ_i, and then a rotation about the z-axis by $\pi/2$. The rotation matrices are [13]

$$\mathbb{R}_y = \begin{pmatrix} C & 0 & -S \\ 0 & 1 & 0 \\ S & 0 & C \end{pmatrix} \qquad \mathbb{R}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{185}$$

giving the total rotation matrix

$$\mathbb{R} = \mathbb{R}_z \cdot \mathbb{R}_y = \begin{pmatrix} 0 & 1 & 0 \\ -C & 0 & S \\ S & 0 & C \end{pmatrix} \qquad \mathbb{R}^{-1} = \begin{pmatrix} 0 & -C & S \\ 1 & 0 & 0 \\ 0 & S & C \end{pmatrix} \tag{186}$$

such that

$$\mathbf{E}' = \mathbb{R}\cdot\mathbf{E} \qquad\qquad \mathbf{E} = \mathbb{R}^{-1}\cdot\mathbf{E}' \tag{187}$$

Now, the transmitted and reflected fields found in special case 1 are the fields in the new primed system for this situation.

incident fields The incident fields in this case are given by equations 88, 89, and 90.

transmitted fields Transmitted fields from the special case 1 unprimed system are now the fields in the primed system

$$\mathcal{H}'_x = \mathcal{H}'^b_x = -\frac{2\mathfrak{n}_b}{\mathfrak{n}_b+1}E'^i_y \qquad\qquad E'_x = E'^a_x = \frac{2}{\mathfrak{n}_a+1}E'^i_x \tag{188}$$

$$\mathcal{H}'_y = \mathcal{H}'^a_y = \frac{2\mathfrak{n}_a}{\mathfrak{n}_a+1}E'^i_x \qquad\qquad E'_y = E'^b_y = \frac{2}{\mathfrak{n}_b+1}E'^i_y \tag{189}$$

$$\mathcal{H}'_z = 0 \qquad\qquad E'_z = E'^b_z = -\frac{i2Y(\mathfrak{n}_b-1)}{1-X}E'^i_y \tag{190}$$

and need to be rotated into the unprimed system via the inverse rotation on the right side of equation 187 resulting in

$$\mathcal{H}_x = \mathcal{H}^a_x = -\frac{2C\mathfrak{n}_a}{\mathfrak{n}_a+1}E'^i_x \qquad\qquad E_x = E^b_x = -\frac{2C}{\mathfrak{n}_b+1}E'^i_y - \frac{i2SY(\mathfrak{n}_b-1)}{1-X}E'^i_y \tag{191}$$

$$\mathcal{H}_y = \mathcal{H}^b_y = -\frac{2\mathfrak{n}_b}{\mathfrak{n}_b+1}E'^i_y \qquad\qquad E_y = E^a_y = \frac{2}{\mathfrak{n}_a+1}E'^i_x \tag{192}$$

$$\mathcal{H}_z = \mathcal{H}^a_z = \frac{2S\mathfrak{n}_a}{\mathfrak{n}_a+1}E'^i_x \qquad\qquad E_z = E^b_z = \frac{2S}{\mathfrak{n}_b+1}E'^i_y - \frac{i2CY(\mathfrak{n}_b-1)}{1-X}E'^i_y \tag{193}$$

Notice that the fields from mode a in the primed system (equations 188, 189, 190) rotate into mode b in the unprimed system (equations 191, 192, 193) and vice-versa.

To fully solve the transmitted fields, the incident fields in the primed system must be expressed in the unprimed system - where they are known. Again, the inverse rotation on the right side of equation 187 is used to get

$$E'^i_x = E^i_y \tag{194}$$

$$E'^i_y = -C^2E^i_\parallel - S^2E^i_\parallel = -E^i_\parallel \tag{195}$$

$$E'^i_z = SCE^i_\parallel - SCE^i_\parallel = 0 \tag{196}$$

The final form of the transmitted fields in the unprimed coordinate system is

$$\mathcal{H}_x = \mathcal{H}^a_x = -\frac{2C\mathfrak{n}_a}{\mathfrak{n}_a+1}E^i_y \qquad\qquad E_x = E^b_x = \frac{2C}{\mathfrak{n}_b+1}E^i_\parallel + \frac{i2SY(\mathfrak{n}_b-1)}{1-X}E^i_\parallel \tag{197}$$

$$\mathcal{H}_y = \mathcal{H}^b_y = \frac{2\mathfrak{n}_b}{\mathfrak{n}_b+1}E^i_\parallel \qquad\qquad E_y = E^a_y = \frac{2}{\mathfrak{n}_a+1}E^i_y \tag{198}$$

$$\mathcal{H}_z = \mathcal{H}^a_z = \frac{2S\mathfrak{n}_a}{\mathfrak{n}_a+1}E^i_y \qquad\qquad E_z = E^b_z = -\frac{2S}{\mathfrak{n}_b+1}E^i_\parallel + \frac{i2CY(\mathfrak{n}_b-1)}{1-X}E^i_\parallel \tag{199}$$

reflected fields Using the same procedure as above, the reflected fields for this case are

$$\mathscr{H}^r_x = -C \cdot_{\parallel} R_{\parallel} \cdot E^i_y \qquad E^r_x = C \cdot_{\perp} R_{\perp} \cdot E^i_{\parallel} \tag{200}$$
$$\mathscr{H}^r_y = -_{\perp} R_{\perp} \cdot E^i_{\parallel} \qquad E^r_y = -_{\parallel} R_{\parallel} \cdot E^i_y \tag{201}$$
$$\mathscr{H}^r_z = S \cdot_{\parallel} R_{\parallel} \cdot E^i_y \qquad E^r_z = -S \cdot_{\perp} R_{\perp} \cdot E^i_{\parallel} \tag{202}$$

where $_{\parallel}R_{\parallel}$ and $_{\perp}R_{\perp}$ are given in equation 165.

5.4.3 case 3: incident wave and magnetic field parallel, oriented perpendicular to interface plane

This case is almost identical to that in section 5.4.1 where the incident, reflected and refracted angles are all zero, except **Y** is exactly parallel (ar antiparallel) to **S**. No coordinate rotation is needed to solve for the transmitted and reflected fields. However, for this case cross coupling from the incident wave fields to each of the refracted modes will exist. These two refracted modes are the R (fast root of the CPDR) and L (slow root of the CPDR) principal modes for EM waves propagating in a magnetized plasma. These modes are EM waves propagating exactly parallel or antiparallel to **Y**, and have no longitudinal component [2, 1, 3]. The wave fields for each mode rotate in a CW (R for Right rotation) or CCW (L for Left rotation) orientation as observed looking along the **B** ($-$**Y**) direction.

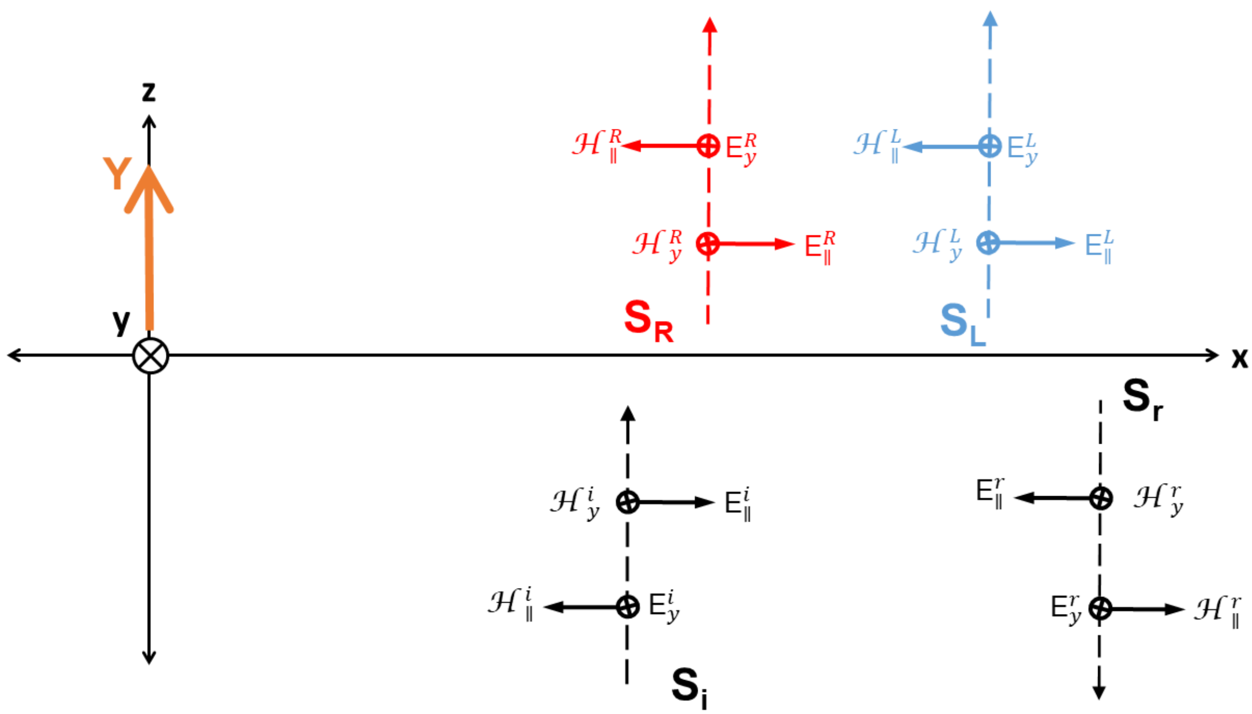

Figure 12: *Perpendicular wave incidence along z with $\theta_i = \theta_r = \theta_{R,L} = 0$. $\mathbf{Y} = Y \cdot (0,0,1)$ and $\mathbf{S} = S \cdot (0,0,1)$. On the plasma side: Red = R mode, Blue = L mode.*

The polarization ratios for these waves are

$$\rho_a = \rho_R = -i \tag{203}$$
$$\rho_b = \rho_L = +i \tag{204}$$
$$Q_{a,b} = Q_{R,L} = 0 \tag{205}$$

which show that the R/L-wave E-fields ($E_y^{R,L}, E_\parallel^{R,L}$) are phased by $\mp\pi/2$ respectively - as expected for circular polarization.

The transmitted and reflected wave field solutions can be found using Cramer's method. Here, the solutions will be found using only the reflection coefficients and algebra. Recall that cross coupling between the incident wave fields and *both* modes can occur. However, the four reflection coefficients are simplified, and given by [4]

$$
\begin{aligned}
{}_\parallel R_\parallel = -{}_\perp R_\perp &= \frac{1}{2}\left[\frac{\mathfrak{n}_R - 1}{\mathfrak{n}_R + 1} + \frac{\mathfrak{n}_L - 1}{\mathfrak{n}_L + 1}\right] && (206)\\
{}_\parallel R_\perp = {}_\perp R_\parallel &= i\left[\frac{1}{\mathfrak{n}_L + 1} - \frac{1}{\mathfrak{n}_R + 1}\right] && (207)
\end{aligned}
$$

Using these, ρ_R, and ρ_L, the incident reflected and transmitted fields can now be specified.

incident fields The incident fields are the same as in section 5.4.1, equations 166, 167, and 168.

$$
\begin{aligned}
\mathscr{H}_x^i &= -E_y^i & E_x^i &= E_\parallel^i && (208)\\
\mathscr{H}_y^i &= E_\parallel^i & E_y^i &= E_y^i && (209)\\
\mathscr{H}_z^i &= 0 & E_z^i &= 0 && (210)
\end{aligned}
$$

reflected fields The reflected wave field components are slightly more complicated than in section 5.4.1, because now one polarization can reflect into the other

$$
\begin{aligned}
\mathscr{H}_x^r &= {}_\parallel R_\perp E_x^i + {}_\perp R_\perp E_y^i & E_x^r &= -{}_\parallel R_\parallel E_x^i - {}_\perp R_\parallel E_y^i && (211)\\
\mathscr{H}_y^r &= {}_\parallel R_\parallel E_x^i + {}_\perp R_\parallel E_y^i & E_y^r &= {}_\parallel R_\perp E_x^i + {}_\perp R_\perp E_y^i && (212)\\
\mathscr{H}_z^r &= 0 & E_z^r &= 0 && (213)
\end{aligned}
$$

transmitted fields The transmitted (refracted) fields are specified using the polarization relations from equation 203, 204, and 205, and Faraday's law from equation 44

$$
\begin{aligned}
\mathscr{H}_x^t &= -\mathfrak{n}_R\rho_R E_x^R - \mathfrak{n}_L\rho_L E_x^L & E_x^t &= E_x^R + E_x^L && (214)\\
\mathscr{H}_y^t &= \mathfrak{n}_R E_x^R + \mathfrak{n}_L E_x^L & E_y^t &= \rho_R E_x^R + \rho_L E_x^L && (215)\\
\mathscr{H}_z^t &= 0 & E_z^t &= 0 && (216)
\end{aligned}
$$

continuity of tangential fields at the interface As before, the incident and reflected field components tangential to the vacuum/plasma interface are equated to their corresponding transmitted fields.

$$E_x \quad \longrightarrow \quad \left(1 - {}_\parallel R_\parallel\right) E_x^i - {}_\perp R_\parallel E_y^i = E_x^R + E_x^L \tag{217}$$

$$E_y \quad \longrightarrow \quad {}_\parallel R_\perp E_x^i + \left(1 + {}_\perp R_\perp\right) E_y^i = \rho_R E_x^R + \rho_L E_x^L \tag{218}$$

$$\mathscr{H}_x \quad \longrightarrow \quad {}_\parallel R_\perp E_x^i + \left({}_\perp R_\perp - 1\right) E_y^i = -\mathfrak{n}_R\rho_R E_x^R - \mathfrak{n}_L\rho_L E_x^L \tag{219}$$

$$\mathscr{H}_y \quad \longrightarrow \quad \left(1 + {}_\parallel R_\parallel\right) E_x^i + {}_\perp R_\parallel E_y^i = \mathfrak{n}_R E_x^R + \mathfrak{n}_L E_x^L \tag{220}$$

This system of four equations and two unknowns is over determined since the reflected fields have been expressed in terms of the incident fields and their reflection coefficients. The unknown fields E_x^R and E_x^L are solved using back substitution from two of the four equations

$$E_x^R = \frac{E_x^i + iE_y^i}{\mathfrak{n}_R + 1} \tag{221}$$

$$E_x^L = \frac{E_x^i - iE_y^i}{\mathfrak{n}_L + 1} \tag{222}$$

from which the complete set of transmitted fields is found

$$\mathscr{H}_x^t = -\left(\frac{\mathfrak{n}_R}{\mathfrak{n}_R + 1} + \frac{\mathfrak{n}_L}{\mathfrak{n}_L + 1}\right) E_y^i + i\left(\frac{\mathfrak{n}_R}{\mathfrak{n}_R + 1} - \frac{\mathfrak{n}_L}{\mathfrak{n}_L + 1}\right) E_x^i \qquad E_x^t = \frac{E_x^i + iE_y^i}{\mathfrak{n}_R + 1} + \frac{E_x^i - iE_y^i}{\mathfrak{n}_L + 1} \tag{223}$$

$$\mathscr{H}_y^t = \left(\frac{\mathfrak{n}_R}{\mathfrak{n}_R + 1} + \frac{\mathfrak{n}_L}{\mathfrak{n}_L + 1}\right) E_x^i + i\left(\frac{\mathfrak{n}_R}{\mathfrak{n}_R + 1} - \frac{\mathfrak{n}_L}{\mathfrak{n}_L + 1}\right) E_y^i \qquad E_y^t = \frac{E_y^i - iE_x^i}{\mathfrak{n}_R + 1} + \frac{E_y^i + iE_x^i}{\mathfrak{n}_L + 1} \tag{224}$$

$$\mathscr{H}_z^t = 0 \qquad E_z^t = 0 \tag{225}$$

6 wave propagation in the ionosphere

Now that the complete set of refracted wave fields has been found, they can be propagated through the ionosphere with a suitably chosen Ionospheric Transfer Function (ITF).

This report does not cover specific algorithms for propagating the refracted waves through the ionosphere, that is covered elsewhere [15, 16, 17], but there are a few items worth noting.

First, while the method presented in this report does not propagate each refracted wave through the ionosphere, the Booker quartic in section 5.1 gives the refracted angles on the plasma side of the vacuum/plasma interface at the bottom of the ionosphere for each wave component. These angles are functions of the incident angle, and serve as the starting point for a refraction algorithm. From this point, the refracted waves can be solved in any number of ways; line of sight, ray trace, and stratified multi-layer to name a few.

Next, the method presented in this report depends on definite boundaries at the bottom and top of the ionosphere. This assumption is the cornerstone for solving the refracted fields at the boundaries, specifically each field's mode (fast/slow) amplitude.

Lastly, since the frequency band of interest for this report is VHF, the wavelengths involved have scale lengths which justify the definite boundary assumption.

7 wave incidence at the top of the ionosphere

Consider a wave $\mathbf{S}_i$ incident at the plasma/vacuum interface on the top of the ionosphere. Here, the wave is incident from inside the plasma, partially transmits into free space, and reflects back into the plasma. The wave will actually reflect into two modes because of the birefringent nature of the magnetized plasma. As before, label the two modes a and b, with the phase velocity of $\mathbf{S}_a$ being

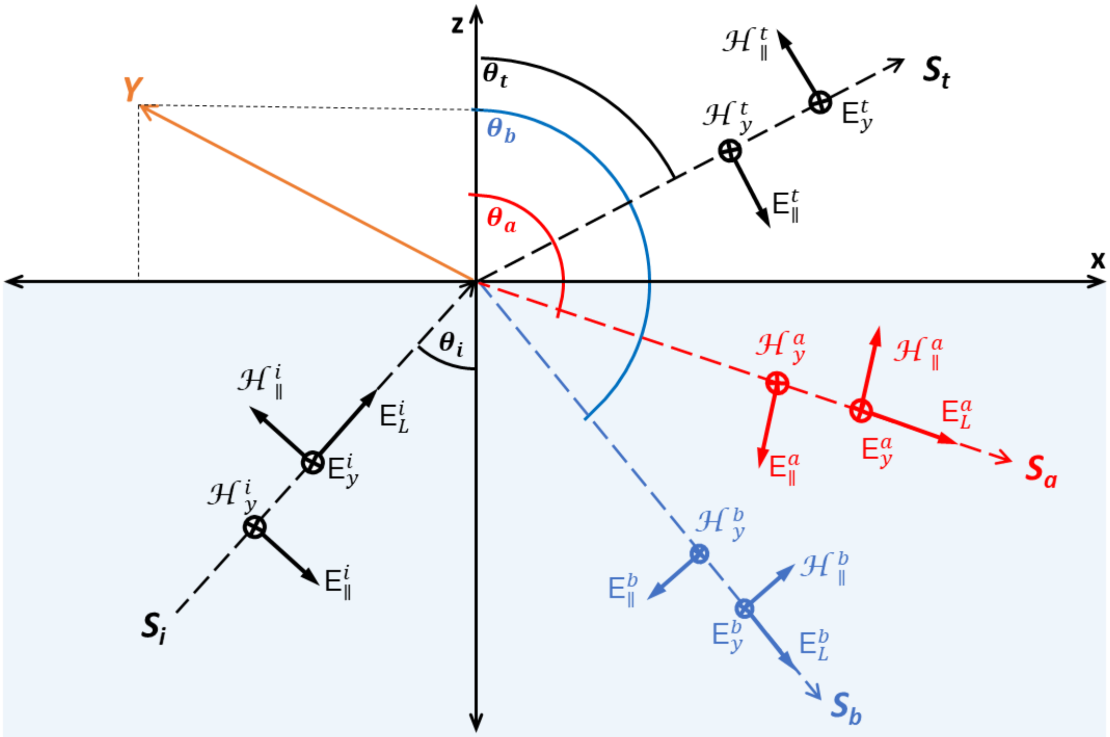

Figure 13: *Geometry for wave and magnetic field co-planar at the top of the ionosphere. Note that the y-coordinate points in to the page, and is different from the vector* $\mathbf{Y}$ *(in orange) which is oblique with direction cosines* $\alpha, \beta, \gamma \neq 0$. *The plasma volume is defined by* $z \leq 0$. *Here, the incident wave reflects into two modes* (a, b) *back into the plasma. The vectors* $\mathbf{Y}$, $\mathbf{S_i}$, *and* $\mathbf{S_j}$ *have direction cosines* $(\cos\alpha, \cos\beta, \cos\gamma)$, $(\sin\theta_i, 0, \cos\theta_i)$, *and* $(\sin\theta_j, 0, \cos\theta_j)$; *where* $j = a, b$.

greater than that of $\mathbf{S}_b$. Actually, the incident wave in this situation is one of the two upward traveling modes that have refracted from an initial EM wave incident on the ionospheric underside, as shown in figure 1. Thus, there will be four reflected, two incident and two transmitted waves expected at the top, as long as neither incident wave has been cut off. It is only necessary to specify the reflected and transmitted wave components for one incident mode reflecting into two modes, as shown in figure 13; addition of the second incident, second transmitted, and third and fourth reflected, modes follows by inspection.

Again, the simpler case of the magnetic meridian plane confined to be along principal axes will be introduced, following with the fully oblique case. Further, matching of tangential fields at the plasma/vacuum boundary will be skipped until the section on fully oblique incidence.

7.1 case I: magnetic meridian plane along principal axes

As before, the incident wave and magnetic field are both constrained to the $x-z$ plane. This situation is shown in figure 13, except that the vector $\mathbf{Y}$ will be in the $x-z$ plane instead of totally oblique. Direction cosines for the vectors are

$$\mathbf{Y} \qquad (l,\ 0,\ n) \tag{226}$$

$$\mathbf{S_i} \qquad (\sin\theta_i,\ 0,\ \cos\theta_i) \tag{227}$$

$$\mathbf{S_a} \qquad (\sin\theta_a,\ 0,\ \cos\theta_a) \tag{228}$$

$$\mathbf{S_b} \qquad (\sin\theta_b,\ 0,\ \cos\theta_b) \tag{229}$$

$$\mathbf{S_t} \qquad (\sin\theta_t,\ 0,\ \cos\theta_t) \tag{230}$$

where now

$$\mathfrak{n}_i \sin\theta_i = \mathfrak{n}_a \sin\theta_a = \mathfrak{n}_b \sin\theta_b \tag{231}$$

Just as in section 5.2, the incident, reflected, and transmitted fields can now be written out. In what follows, as in section 5.2, Faraday's law, equation 87, is used to relate the $\mathscr{H}$ and E fields.

7.1.1 incident fields

The components of the incident wave fields are

$$\begin{aligned} \mathscr{H}_x^i &= -\mathscr{H}_\parallel^i \cos\theta_i & E_x^i &= E_L^i \sin\theta_i + E_\parallel^i \cos\theta_i \\ &= -\mathfrak{n}_i E_y^i \cos\theta_i \end{aligned} \tag{232}$$

$$\begin{aligned} \mathscr{H}_y^i &= \mathscr{H}_y^i & E_y^i &= E_y^i \\ &= \mathfrak{n}_i E_\parallel^i \end{aligned} \tag{233}$$

$$\begin{aligned} \mathscr{H}_z^i &= \mathscr{H}_\parallel^i \sin\theta_i & E_z^i &= E_L^i \cos\theta_i - E_\parallel^i \sin\theta_i \\ &= -\mathfrak{n}_i E_y^i \sin\theta_i \end{aligned} \tag{234}$$

7.1.2 reflected fields

There will be two sets of fields in this case because the wave will reflect back into the plasma. They will represent the two birefringent modes a, b present in a magnetized plasma. For simplicity, let

$$\phi_j = \pi - \theta_j \tag{235}$$

where $j = a, b$. Then the reflected fields are

$$\begin{aligned} \mathscr{H}_x^j &= \mathscr{H}_\parallel^j \cos\phi_j & E_x^j &= E_L^j \sin\phi_j + E_\parallel^j \cos\phi_j \\ &= \mathfrak{n}_j E_y^j \cos\phi_a \end{aligned} \tag{236}$$

$$\begin{aligned} \mathscr{H}_y^j &= \mathscr{H}_y^j & E_y^j &= E_y^j \\ &= \mathfrak{n}_j E_\parallel^j \end{aligned} \tag{237}$$

$$\begin{aligned} \mathscr{H}_z^j &= \mathscr{H}_\parallel^j \sin\phi_j & E_z^j &= -E_L^j \cos\phi_j - E_\parallel^j \sin\phi_j \\ &= -\mathfrak{n}_j E_y^j \sin\phi_j \end{aligned} \tag{238}$$

These field components can be further reduced using the polarization relations in equations 94, but this is not necessary since the full wave field component solutions will not be found until the next section when fully oblique incidence is considered.

7.1.3 transmitted fields

In this situation, the transmitted fields are in vacuum, and can be written as

$$\begin{aligned} \mathscr{H}_x^t &= -\mathscr{H}_\parallel^t \cos\theta_t & \qquad E_x^t &= E_\parallel^t \cos\theta_t \\ &= -E_y^i \cos\theta_t \end{aligned} \tag{239}$$

$$\begin{aligned} \mathscr{H}_y^t &= \mathscr{H}_y^t & \qquad E_y^t &= E_y^t \\ &= E_\parallel^t \end{aligned} \tag{240}$$

$$\begin{aligned} \mathscr{H}_z^t &= \mathscr{H}_\parallel^t \sin\theta_t & \qquad E_z^i &= E_L^i \cos\theta_i - E_\parallel^i \sin\theta_i \\ &= -E_y^t \sin\theta_t \end{aligned} \tag{241}$$

7.2 case II: magnetic meridian plane oblique (general case)

This case is much like the situation in section 5.3, except now the wave is incident from inside the plasma and will reflect back in to the plasma.

7.2.1 orientation at the interface

The situation is shown in figure 14. As before, it is desirable to rotate into a new coordinate system

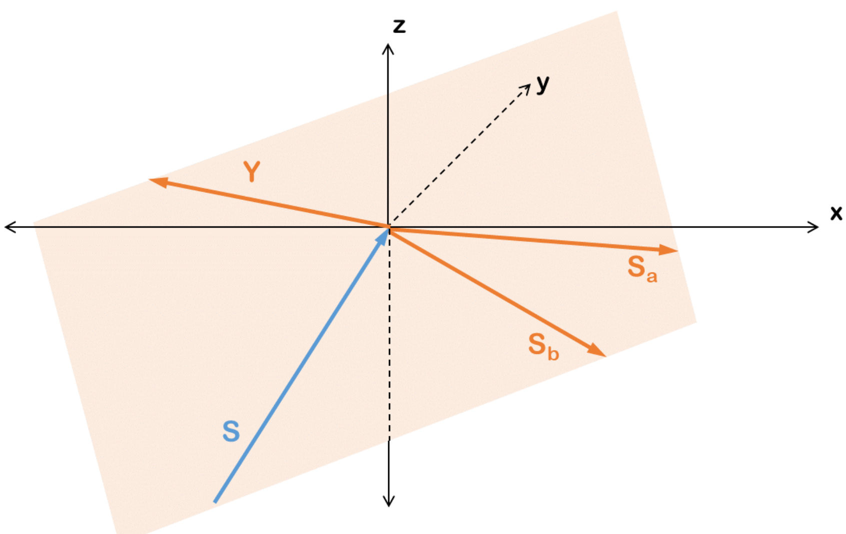

Figure 14: *The orientation for oblique incident and refracted waves in a collisionless, magnetized plasma.* **S** *is incident from below the plasma/vacuum interface* ($x-y$ *plane) - from* inside *the plasma. It will reflect into two modes* $\mathbf{S_a}$ *and* $\mathbf{S_b}$ *(assuming neither is cutoff) in the* $\mathbf{S}-\mathbf{Y}$ *plane.*

to facilitate calculation of the wave field solutions. The coordinate rotation is shown in figure 15.

Orientation at the point of incidence can always be rotated into the geometry of figure 15(a), just like in section 5.3. However, as before, it is advantageous to rotate into the primed coordinate system geometry of figure 15(b), solve the wave fields, and then apply the inverse coordinate rotation to get the solutions in the unprimed coordinate system.

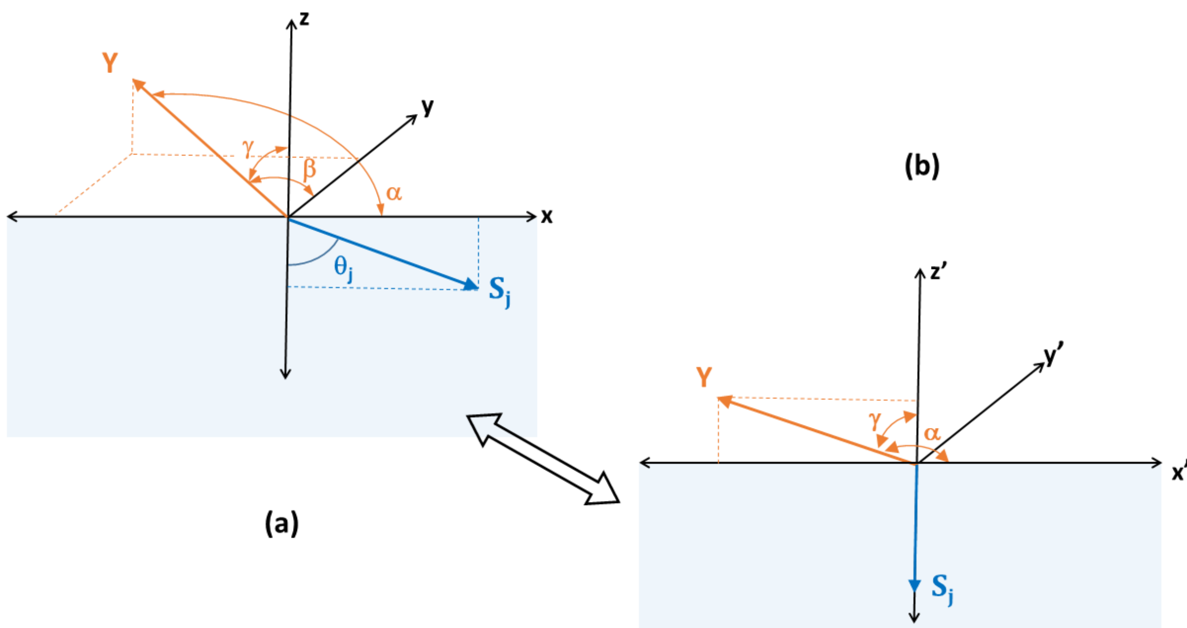

Figure 15: *Coordinate rotation from oblique unprimed system (a) to final primed system (b). The unprimed direction cosines* (l, m, n) *are* $(\cos\alpha, \cos\beta, \cos\gamma)$ *and the reflected angle* θ_j *is known, where j represents the reflected mode a or b. In the primed coordinate system,* (l, m, n) *are* $(\cos\alpha, 0, \cos\beta)$*, and* $\theta_j = 0$*. The direction cosines for* $\mathbf{S_j}$ *in the unprimed coordinate system are* $(l, m, n) = (\sin\theta_j, 0, -\cos\theta_j.)$

7.2.2 coordinate rotation matrix

The coordinate rotation matrix is found in the same manner as it was in section 5.3.2. The only difference is that the primed coordinate system will be oriented such that $\hat{z}'$ will be *antiparallel* to $\mathbf{S_j}$ (compare to equation 105)

$$\hat{a}_3 = -\frac{\mathbf{S_j}}{|\mathbf{S_j}|} = (-S_j, 0, -C_j) \tag{242}$$

where $S_j = \sin\theta_j, C_j = \cos\theta_j$. That is, the modes to be solved in the plasma are now the reflected modes. Also, as before, there will be a rotation matrix *for each mode.*

With the above information, the coordinate rotation matrix to go from the unprimed to primed coordinate system in figure 15 is

$$\mathbb{M}_j = \begin{bmatrix} \dfrac{l\cos\theta_j - n\sin\theta_j}{G_j}\cos\theta_j & \dfrac{m}{G_j} & \dfrac{l\cos\theta_j - n\sin\theta_j}{G_j}(-\sin\theta_j) \\ \dfrac{m\cos\theta_j}{G_j} & -\dfrac{l\cos\theta_j - n\sin\theta_j}{G_j} & -\dfrac{m\sin\theta_j}{G_j} \\ -\sin\theta_j & 0 & -\cos\theta_j \end{bmatrix} \tag{243}$$

where (l, m, n) are the direction cosines of $\mathbf{Y}$ and $j = a, b$, and G is defined in equation 107.

The fields in the primed coordinate system are shown in figure 16.

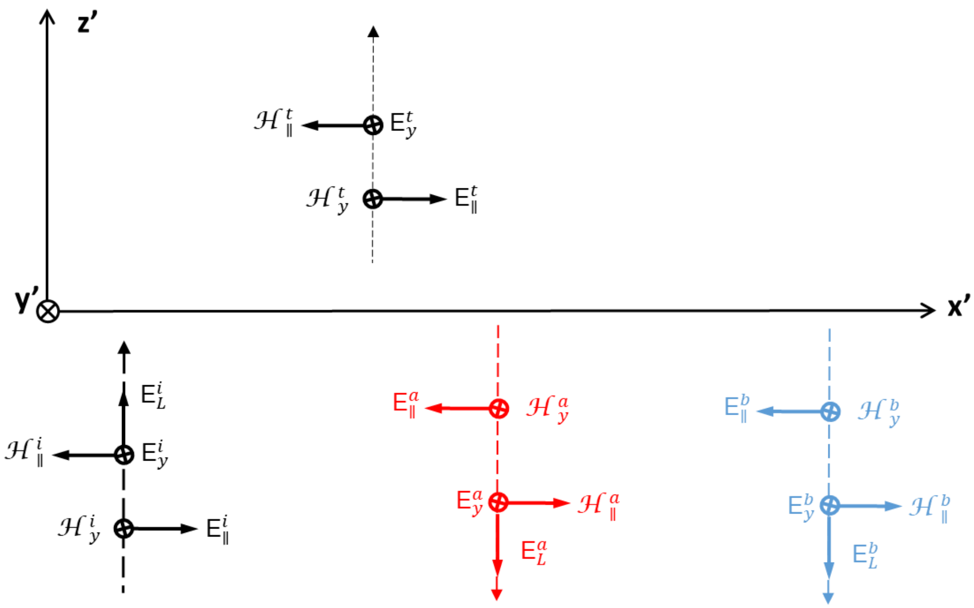

Figure 16: *Fields at the topside plasma/vacuum interface - primed coordinate system.*

Following the method in sections 5.3.3, 5.3.4, and 5.3.5, start in the primed coordinate system and work back to the unprimed coordinate system. It is evident from those sections that only the reflected fields need to be specified at this point, since their solutions directly involve the Booker quartic and polarization relations in the plasma. After they are specified, they can be rotated out to the unprimed system and their tangential fields equated with those of the incident and transmitted fields.

7.2.3 reflected fields

The reflected fields in this situation will be a superposition of the two birefringent modes in the plasma. These fields can be written using the polarization relations from equation 94 along with the Booker quartic to solve the different refractive indices. However, in this case, choose the solutions with a *positive* imaginary part since the reflected waves will be traveling downward.

$$\mathscr{H}^j_{x'} = \mathfrak{n}_j E^j_{y'} \qquad\qquad E^j_{x'} = \frac{-1}{\rho_j} E^j_{y'} \tag{244}$$

$$\mathscr{H}^j_{y'} = \frac{\mathfrak{n}_j}{\rho_j} E^j_{y'} \qquad\qquad E^j_{y'} = E^j_{y'} \tag{245}$$

$$\mathscr{H}^j_{z'} = 0 \qquad\qquad E^j_{z'} = -Q_j E^j_{y'} \tag{246}$$

where $j = a, b$ and $\mathfrak{n}_i$ is the index of refraction for the incident wave. Recall that Q_j, ρ_j, and $\mathfrak{n}_j$ are functions of the local plasma parameters only (see table 1), which are at the ionosphere topside for this case. Notice also that a few steps have been skipped by using the polarization relations allowing the reflected fields to be written in terms of $E^j_{y'}$.

The next step is to rotate these fields to the unprimed coordinate system via the inverse of the coordinate rotation matrix in equation 243. Let

$$\mathbb{m}_j = \mathbb{M}_j^{-1}$$

Then

$$\mathbf{E}^j = \mathbb{m}_j \cdot \mathbf{E}'^j$$

and the reflected wave fields in the unprimed coordinate system are

$$E^a_x = \left[-\mathbb{m}^a_{11}\frac{1}{\rho_a} + \mathbb{m}^a_{12} - \mathbb{m}^a_{13} Q_a\right] E'^a_y \qquad = A_1 E'^a_y \tag{247}$$

$$E^b_x = \left[-\mathbb{m}^b_{11}\frac{1}{\rho_b} + \mathbb{m}^b_{12} - \mathbb{m}^b_{13} Q_b\right] E'^b_y \qquad = B_1 E'^b_y \tag{248}$$

$$E^a_y = \left[\mathbb{m}^a_{21}\frac{1}{\rho_a} + \mathbb{m}^a_{22} - \mathbb{m}^a_{23} Q_a\right] E'^a_y \qquad = A_2 E'^a_y \tag{249}$$

$$E^b_y = \left[\mathbb{m}^b_{21}\frac{1}{\rho_b} + \mathbb{m}^b_{22} - \mathbb{m}^b_{23} Q_b\right] E'^b_y \qquad = B_2 E'^b_y \tag{250}$$

$$E^a_z = \left[\mathbb{m}^a_{31}\frac{1}{\rho_a} + \mathbb{m}^a_{32} - \mathbb{m}^a_{33} Q_a\right] E'^a_y \qquad = A_3 E'^a_y \tag{251}$$

$$E^b_z = \left[\mathbb{m}^b_{31}\frac{1}{\rho_b} + \mathbb{m}^b_{32} - \mathbb{m}^b_{33} Q_b\right] E'^b_y \qquad = B_3 E'^b_y \tag{252}$$

$$\mathscr{H}_x^a = \left[\mathfrak{n}_a \mathrm{m}_{11}^a \;+\; \frac{\mathfrak{n}_a}{\rho_a} \mathrm{m}_{12}^a\right] E'^a_y \qquad = A_4 E'^a_y \tag{253}$$

$$\mathscr{H}_x^b = \left[\mathfrak{n}_b \mathrm{m}_{11}^b \;+\; \frac{\mathfrak{n}_b}{\rho_b} \mathrm{m}_{12}^b\right] E'^b_y \qquad = B_4 E'^b_y \tag{254}$$

$$\mathscr{H}_y^a = \left[\mathfrak{n}_a \mathrm{m}_{21}^a \;+\; \frac{\mathfrak{n}_a}{\rho_a} \mathrm{m}_{22}^a\right] E'^a_y \qquad = A_5 E'^a_y \tag{255}$$

$$\mathscr{H}_y^b = \left[\mathfrak{n}_b \mathrm{m}_{21}^b \;+\; \frac{\mathfrak{n}_b}{\rho_b} \mathrm{m}_{22}^b\right] E'^b_y \qquad = B_5 E'^b_y \tag{256}$$

$$\mathscr{H}_z^a = \left[\mathfrak{n}_a \mathrm{m}_{31}^a \;+\; \frac{\mathfrak{n}_a}{\rho_a} \mathrm{m}_{32}^a\right] E'^a_y \qquad = A_6 E'^a_y \tag{257}$$

$$\mathscr{H}_z^b = \left[\mathfrak{n}_b \mathrm{m}_{31}^b \;+\; \frac{\mathfrak{n}_b}{\rho_b} \mathrm{m}_{32}^b\right] E'^b_y \qquad = B_6 E'^b_y \tag{258}$$

It is not necessary to write out the incident wave fields, as they are known in the unprimed coordinate system.

7.2.4 transmitted fields

The transmitted fields in vacuum are

$$\begin{aligned} \mathscr{H}_x^t &= -\mathscr{H}_\parallel^i \cos\theta_t \qquad\qquad E_x^i = E_\parallel^t \cos\theta_t \\ &= -E_y^t \cos\theta_t \end{aligned} \tag{259}$$

$$\begin{aligned} \mathscr{H}_y^t &= \mathscr{H}_y^t \qquad\qquad E_y^t = E_y^t \\ &= E_\parallel^t \end{aligned} \tag{260}$$

$$\begin{aligned} \mathscr{H}_z^t &= \mathscr{H}_\parallel^t \sin\theta_t \qquad\qquad E_z^t = -E_\parallel^t \sin\theta_t \\ &= -E_y^t \sin\theta_t \end{aligned} \tag{261}$$

and the angle of transmission with respect to the plasma/vacuum interface vertical ($\hat{z}$) is given by Snell's law

$$\theta_t = \sin^{-1}\left(\mathfrak{n}_i \sin\theta_i\right) \tag{262}$$

7.2.5 continuity of tangential fields at the interface

Now the tangential wave field components can be equated at the plasma/vacuum interface in the unprimed coordinate system

$$E_x^i + A_1 E_{y'}^a + B_1 E_{y'}^b = E_\parallel^t \cos\theta_t \tag{263}$$

$$E_y^i + A_2 E_{y'}^a + B_2 E_{y'}^b = E_y^t \tag{264}$$

$$\mathscr{H}_x^i + A_4 E_{y'}^a + B_4 E_{y'}^b = -E_y^t \cos\theta_t \tag{265}$$

$$\mathscr{H}_y^i + A_5 E_{y'}^a + B_5 E_{y'}^b = E_\parallel^t \tag{266}$$

This system of four equations in four unknowns is written as

$$\begin{bmatrix} -C & 0 & A_1 & B_1 \\ 0 & -1 & A_2 & B_2 \\ 0 & C & A_4 & B_4 \\ -1 & 0 & A_5 & B_5 \end{bmatrix} \cdot \begin{bmatrix} E_\parallel^t \\ E_y^t \\ E'^a_y \\ E'^b_y \end{bmatrix} = \begin{bmatrix} -E_x^i \\ -E_y^i \\ -\mathscr{H}_x^i \\ -\mathscr{H}_y^i \end{bmatrix} \tag{267}$$

where for this situation, $C = \cos\theta_t$.

Using Cramer's rule, the fields $E_\parallel^t, E_y^t, E_{y'}^a$, and $E_{y'}^b$ can be found.

$$\begin{aligned} E_\parallel^t = \frac{1}{\Delta}\Big[&-CE_y^i(A_1B_5 - A_5B_1) + E_x^i(A_2B_5C - A_5B_2C + A_4B_5 - A_5B_4) \\ &-\mathscr{H}_x^i(A_1B_5 - A_5B_1) + \mathscr{H}_y^i(A_1B_2C - A_2B_1C + A_1B_4 - A_4B_1)\Big] \end{aligned} \tag{268}$$

$$\begin{aligned} E_y^t = \frac{1}{\Delta}\Big[&E_y^i(A_1B_4 - A_4B_1 + A_4B_5C - A_5B_4C) - E_x^i(A_2B_4 - A_4B_2) \\ &+\mathscr{H}_x^i(A_1B_2 - A_2B_1 + A_2B_5C - A_5B_2C) + C\mathscr{H}_y^i(A_2B_4 - A_4B_2)\Big] \end{aligned} \tag{269}$$

$$\begin{aligned} E'^a_y = \frac{1}{\Delta}\Big[&CE_y^i(B_1 - B_5C) - E_x^i(B_2C + B_4) \\ &+\mathscr{H}_x^i(B_1 - B_5C) + C\mathscr{H}_y^i(B_2C + B_4)\Big] \end{aligned} \tag{270}$$

$$\begin{aligned} E'^b_y = \frac{1}{\Delta}\Big[&-CE_y^i(A_1 - A_5C) + E_x^i(A_2C + A_4) \\ &-\mathscr{H}_x^i(A_1 - A_5C) - C\mathscr{H}_y^i(A_2C + A_4)\Big] \end{aligned} \tag{271}$$

$$\Delta = C^2 (A_2 B_5 - B_2 A_5) + C (A_4 B_5 - A_5 B_4 + A_1 B_2 - A_2 B_1) - B_1 A_4 + A_1 B_4 \tag{272}$$

The transmitted wave fields are given by equations 268 and 269, and the angle θ_t from equation 262. These fields can then be propagated to the sensor located above the ionosphere in free space.

7.3 special cases

Just as in section 5.4, There are certain situations at the topside where no coordinates rotation is necessary, or where some members of the matrix $\mathbb{M}_j$ are infinite. They are the same three cases as for the underside, but solution of the wave fields different because the incident wave comes from the plasma side and travels to vacuum.

7.3.1 case I: incident wave perpendicular and magnetic field horizontal at the interface plane

In this case, by definition, the incident wave is a combination of O and X modes and cross coupling between the modes from incident to reflected waves is not possible. The situation is shown in figure 17.

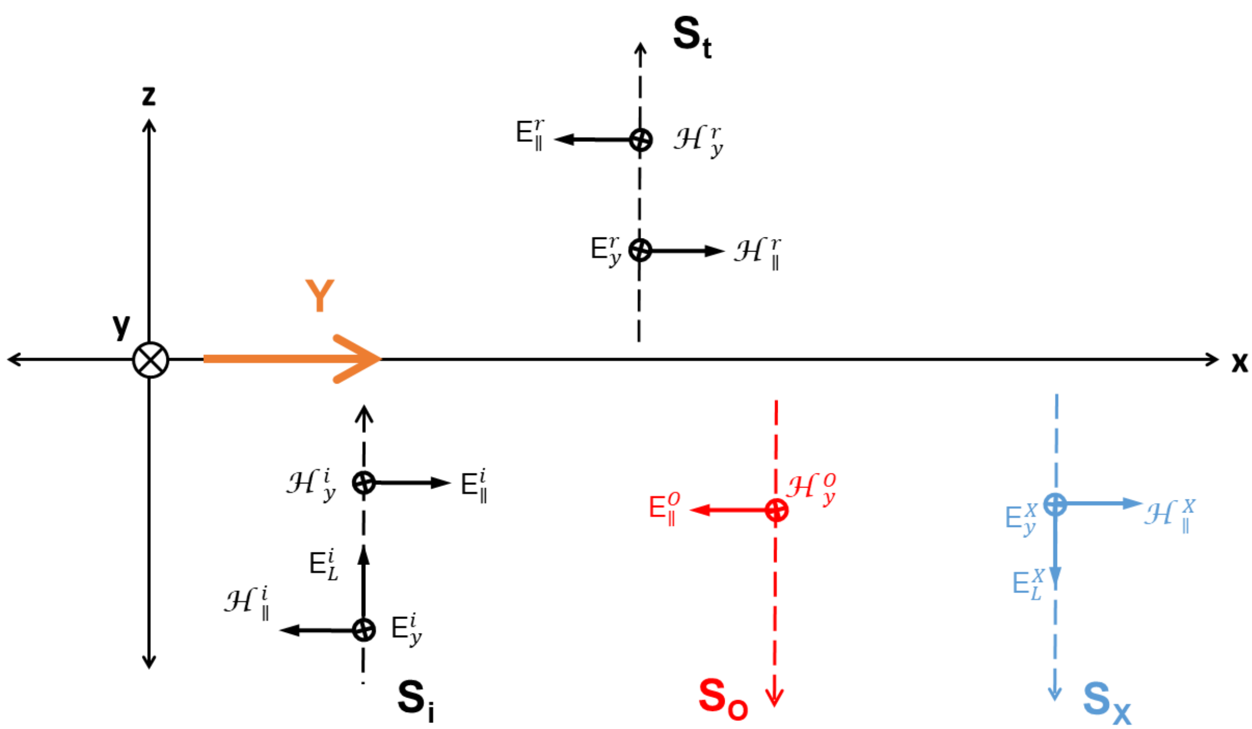

Figure 17: *Perpendicular wave incidence along $\hat{z}$ with $\theta_i = \theta_J = \theta_t = 0$. $\mathbf{Y} = Y \cdot (1,0,0)$ and $\mathbf{S}_J = S \cdot (0,0,-1)$. $j = a,b$; $a \rightarrow O$ mode, $b \rightarrow X$ mode. On the plasma side: Red = O mode, Blue = X mode.*

The Fresnel formulae for reflected and transmitted wave fields can be used in this case, and in fact facilitate solution of the wave fields by inspection. The transmitted and reflected field coefficients for a wave incident from the plasma side and traveling to the vacuum side are [4]

$$ {}_{\parallel}R_{\parallel} = \frac{1-\mathfrak{n}_O}{1+\mathfrak{n}_O} \qquad {}_{\parallel}T_{\parallel} = \frac{2\mathfrak{n}_O}{1+\mathfrak{n}_O} \tag{273} $$

$$ {}_{\perp}R_{\perp} = \frac{\mathfrak{n}_X-1}{\mathfrak{n}_X+1} \qquad {}_{\perp}T_{\perp} = \frac{2\mathfrak{n}_X}{\mathfrak{n}_X+1} \tag{274} $$

The reflected and transmitted field solutions are now written in terms of the incident fields without the need for equating the tangential components at their interface.

reflected fields

$$ E_x^r = -{}_{\parallel}R_{\parallel}E_x^i \qquad \text{O mode} \tag{275} $$

$$ \mathscr{H}_y^r = {}_{\parallel}R_{\parallel}\mathfrak{n}_O E_x^i \tag{276} $$

$$ E_y^r = {}_{\perp}R_{\perp}E_y^i \qquad \text{X mode} \tag{277} $$

$$ E_z^r = -Q_X\,{}_{\perp}R_{\perp}E_y^i \tag{278} $$

$$ \mathscr{H}_x^r = \mathfrak{n}_X\,{}_{\perp}R_{\perp}E_r^i \tag{279} $$

$$ \mathscr{H}_z^r = 0 \tag{280} $$

transmitted fields

$$ E_x^t = -{}_{\parallel}T_{\parallel}E_x^i \qquad \text{O mode} \tag{281} $$

$$ \mathscr{H}_y^r = {}_{\parallel}T_{\parallel}E_x^i \tag{282} $$

$$ E_y^t = {}_{\perp}T_{\perp}E_y^i \qquad \text{X mode} \tag{283} $$

$$ E_z^t = 0 \tag{284} $$

$$ \mathscr{H}_x^t = -{}_{\perp}T_{\perp}E_y^i \tag{285} $$

$$ \mathscr{H}_z^t = 0 \tag{286} $$

The polarization relation Q_X is used in equation 278 to solve E_z for the reflected X mode wave.

7.3.2 case II: incident wave oblique and magnetic field horizontal at the interface plane, perpendicular to incidence plane

The orientation here is the same as in section 5.4.2. However, the plasma is now in the half-space $\hat{z} \leq 0$, there are two reflected modes, and **Y** lies along the $\hat{y}$ axis. This is illustrated in figure 18.

the incident wave is oblique in a plane perpendicular to the magnetic field vector, which lies in the interface plane. It is comprised of one, or a combination of, pure O and X modes, and will reflect into pure O and X modes with no cross coupling. This is basically the same situation as the first special case, where the plane of incidence is the **S**−**Y** plane, except that (just as in section 5.4.2) two coordinate rotations separate this geometry from that in the first special case. Solutions for the transmitted and reflected wave fields can be found by rotating this system into the system from special case 1 and

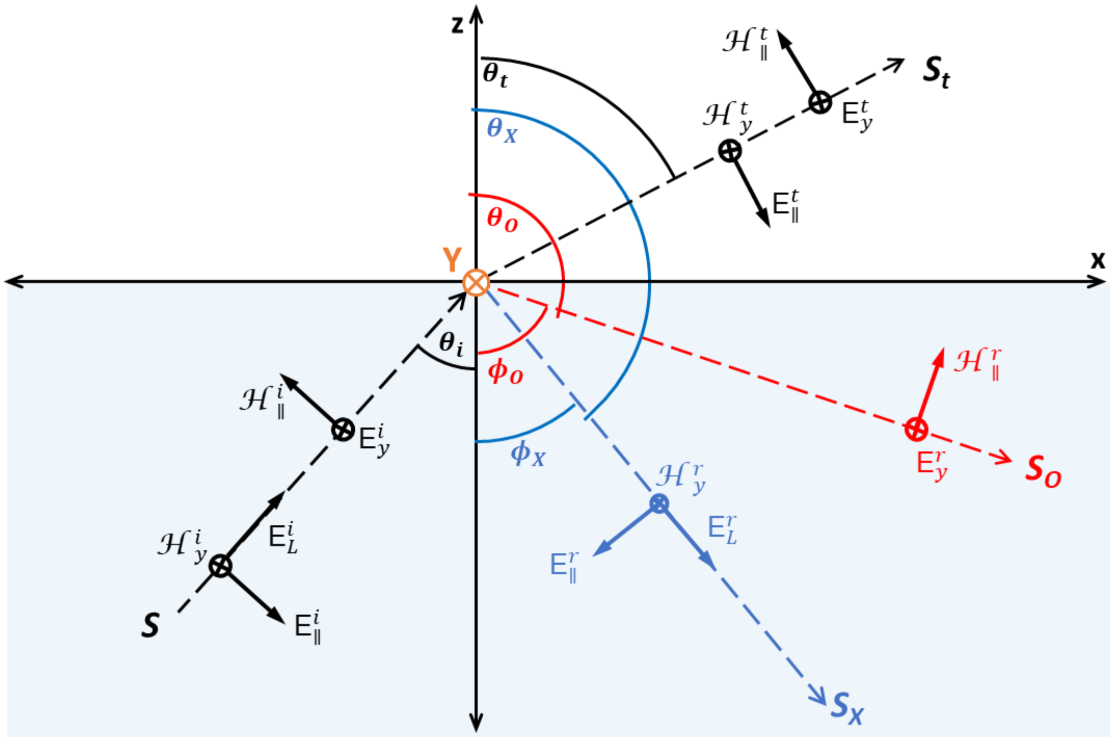

Figure 18: *Oblique wave incidence in the $x-z$ plane.* $\mathbf{Y} = Y\cdot(0,1,0)$, $\mathbf{S_O} = S_O\cdot\left(\sin\phi_O, 0, -\cos\phi_O\right)$, *and* $\mathbf{S_X} = S_X\cdot\left(\sin\phi_X, 0, -\cos\phi_X\right)$.

using those field solutions. Here, the primed coordinate system will be the coordinate system from special case 1. The rotation matrix for this transform is found as in section 5.4.2, but will be slightly different because the reflected modes point in the negative $\hat{z}'$ direction.

To rotate in to the primed coordinate system, first rotate about the $\hat{y}$ axis by $-\phi_j$ and the about the $\hat{z}$ axis by $\pi/2$, where $j = O, X$. This results in

$$\mathbb{R} = \mathbb{R}_z\cdot\mathbb{R}_y = \begin{pmatrix} 0 & 1 & 0 \\ -C_j & 0 & -S_j \\ -S_j & 0 & C_j \end{pmatrix} \qquad \mathbb{R}^{-1} = \begin{pmatrix} 0 & -C_j & -S_j \\ 1 & 0 & 0 \\ 0 & -S_j & C_j \end{pmatrix} \tag{287}$$

such that

$$\mathbf{E}' = \mathbb{R}\cdot\mathbf{E} \qquad \mathbf{E} = \mathbb{R}^{-1}\cdot\mathbf{E}' \tag{288}$$

In the above equations $S_j = \sin\phi_j$, $C_j = \cos\phi_j$, $\phi_j = \pi - \theta_j$, and $j = O, X$. Furthermore, $\mathbb{R}$ will contain S_O, C_O for O mode and S_X, C_X for X mode field rotations.

The wave fields can now be specified, and Cramer's method used to solve them.

reflected fields The reflected fields are found by simply rotating them out from the primed coordinate system using the right equation of 288 and accounting for the mode.

$$E_x^r = -C_X E_{y'}^r - S_X E_{z'}^r \qquad = -(C_x + S_X Q_X)\, E_{y'}^r \qquad \text{X mode} \tag{289}$$

$$E_z^r = -S_X E_{y'}^r + C_X E_{z'}^r \qquad = (C_X Q_X - S_X)\, E_{y'}^r \tag{290}$$

$$\mathscr{H}_y^r = \mathscr{H}_{x'}^r \qquad = \mathfrak{n}_X E_{y'}^r \tag{291}$$

$$E_y^r = E_{x'}^r \qquad = -E_{\|'}^r \qquad \text{O mode} \tag{292}$$

$$\mathscr{H}_x^r = -C_O \mathscr{H}_{y'}^r \qquad = -C_O \mathfrak{n}_O E_{\|'}^r \tag{293}$$

$$\mathscr{H}_z^r = -S_O \mathscr{H}_{y'}^r \qquad = -S_O \mathfrak{n}_O E_{\|'}^r \tag{294}$$

transmitted fields The transmitted fields in the unprimed system are

$$\begin{aligned} \mathscr{H}_x^t &= -\mathscr{H}_{\|} C_t & E_x^t &= E_{\|}^t C_t \\ &= -E_y^y C_t \end{aligned} \tag{295}$$

$$\mathscr{H}_y^t = E_{\|}^t \qquad E_y^t = E_y^t \tag{296}$$

$$\begin{aligned} \mathscr{H}_z^t &= \mathscr{H}_{\|}^t S_t & E_Z^t &= E_{\|}^t S_t \\ &= E_y^t S_t \end{aligned} \tag{297}$$

where S_t, C_t are $\sin\theta_t$ and $\cos\theta_t$.

continuity of tangential field components at the interface The system of four equations in four unknowns found from applying Snell's law to the tangential fields at the interface is

$$\begin{bmatrix} -(C_X + S_X Q_x) & 0 & 0 & -C_t \\ 0 & -1 & -1 & 0 \\ 0 & -C\mathfrak{n}_O & C_t & 0 \\ \mathfrak{n}_X & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} E_{y'}^r \\ E_{\|'}^r \\ E_y^t \\ E_{\|}^t \end{bmatrix} = \begin{bmatrix} -E_x^i \\ -E_y^i \\ \mathfrak{n}_i C_i E_y^i \\ -\mathfrak{n}_i (C_i E_x^i - S_i E_z^i) \end{bmatrix} \tag{298}$$

where S_i, C_i are $\sin\theta_i, \cos\theta_i$ and $\mathfrak{n}_i$ is the refractive index for the incident mode which is either O or X depending on the wave field orientation with respect to **Y**. $\mathscr{H}_x^i$ and $\mathscr{H}_y^i$ were found from geometry and Faraday's law

$$\mathscr{H}_x^i = \mathfrak{n}_i C_i E_y^i \tag{299}$$

$$\mathscr{H}_y^i = -\mathfrak{n}_i (C_i E_x^i - S_i E_z^i) \tag{300}$$

The transmitted fields are

$$\mathscr{H}_x^t = -\frac{C_t E_y^i \left(C_i \mathfrak{n}_i + C_O \mathfrak{n}_O\right)}{C_O \mathfrak{n}_O + C_t} \tag{301}$$

$$\mathscr{H}_y^t = \frac{C_i E_x^i \mathfrak{n}_i \left(C_X + Q_X S_X\right) + E_x^i \mathfrak{n}_X - E_z^i S_i \mathfrak{n}_i \left(C_X + Q_X S_X\right)}{C_t \mathfrak{n}_X + C_X + Q_X S_X} \tag{302}$$

$$\mathscr{H}_z^t = \frac{E_y^i S_t \left(C_i \mathfrak{n}_i + C_O \mathfrak{n}_O\right)}{C_O \mathfrak{n}_O + C_t} \tag{303}$$

$$E_x^t = \frac{C_t \left(C_i E_x^i \mathfrak{n}_i \left(C_X + Q_X S_X\right) + E_x^i \mathfrak{n}_X - E_z^i S_i \mathfrak{n}_i \left(C_X + Q_X S_X\right)\right)}{C_t \mathfrak{n}_X + C_X + Q_X S_X} \tag{304}$$

$$E_y^t = \frac{E_y^i \left(C_i \mathfrak{n}_i + C_O \mathfrak{n}_O\right)}{C_O \mathfrak{n}_O + C_t} \tag{305}$$

$$E_z^t = -\frac{S_t \left(C_i E_x^i \mathfrak{n}_i \left(C_X + Q_X S_X\right) + E_X^i \mathfrak{n}_X - E_z^i S_i \mathfrak{n}_i \left(C_X + Q_X S_X\right)\right)}{C_t \mathfrak{n}_X + C_X + Q_X S_X} \tag{306}$$

7.3.3 case III: incident wave and magnetic field parallel, oriented perpendicular to the interface plane

Here the incident, reflected and transmitted waves are all parallel to the magnetic field, which is perpendicular to the interface plane such that $\theta_i = \theta_{a,b} = \theta_t = 0$. The situation is shown in figure 19. Furthermore, as pointed out in section 5.4.3, The incident wave will be a combination of the R and L modes. In this case, however, it will reflect into R and L modes with the possibility of cross coupling between modes.

In this case, as before, the mode designations will be $a \to R$ and $b \to L$.

$$\mathfrak{n}_a \to \mathfrak{n}_r \tag{307}$$

$$\mathfrak{n}_b \to \mathfrak{n}_L \tag{308}$$

The reflected modes will be traveling in the $-\hat{z}$ direction, so the polarization relations in that direction are

$$\rho_a \to \rho_R = +i \tag{309}$$

$$\rho_b \to \rho_L = -i \tag{310}$$

Also, by definition there will be no E and $\mathscr{H}$ fields in the $\hat{z}$ direction. The subscrupts $\parallel$ and $\hat{x}$ are interchangeable here.

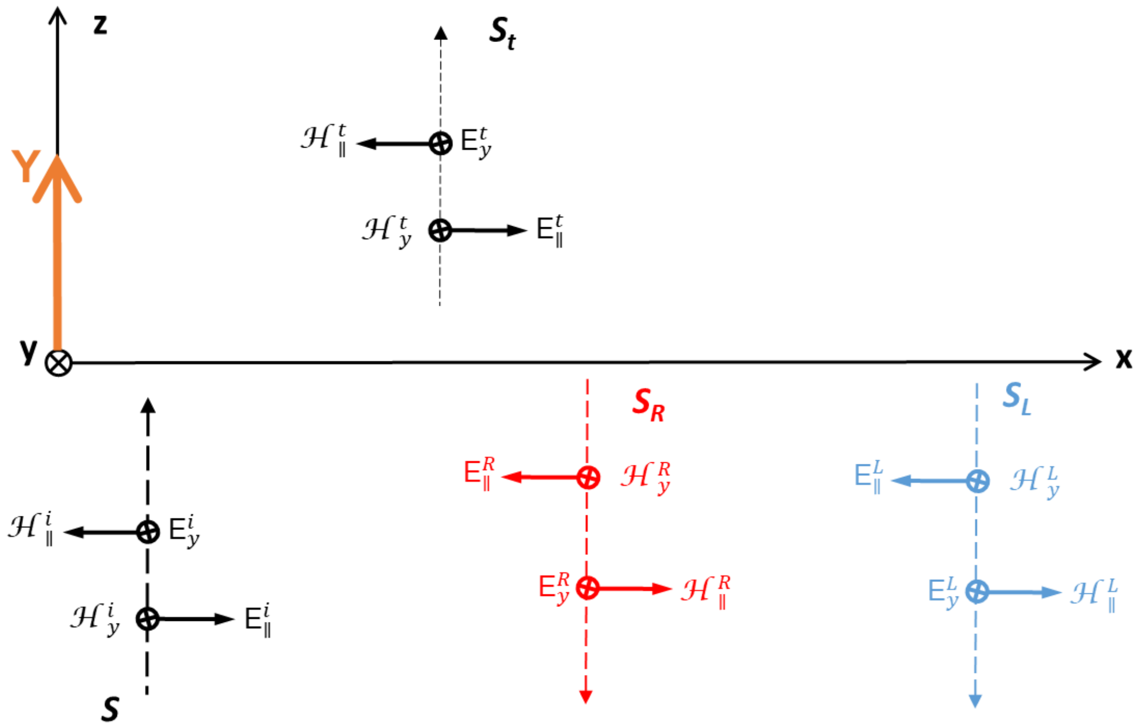

Figure 19: *Perpendicular wave incidence along z with* $\theta_i = \theta_r = \theta_{R,L} = 0$. $\mathbf{Y} = Y \cdot (0,0,1)$ *and* $\mathbf{S}_{R,L} = S_{R,L} \cdot (0,0,-1)$. *On the plasma side: Red = R mode, Blue = L mode.*

incident fields The incident fields are

$$\mathscr{H}_x^i = -\mathfrak{n}_i E_y^i \qquad E_x^i = E_x^i \tag{311}$$

$$\mathscr{H}_y^i = \mathfrak{n}_i E_x^i \qquad E_y^i = E_y^i \tag{312}$$

reflected fields These will be a combination of both modes.

$$\mathscr{H}_x^r = \mathfrak{n}_a \rho_a E_x^a + \mathfrak{n}_b \rho_b E_x^b \qquad E_x^r = -E_x^a - E_x^b \tag{313}$$

$$\mathscr{H}_y^r = \mathfrak{n}_a E_x^a + \mathfrak{n}_b E_x^b \qquad E_y^r = \rho_a E_x^a + \rho_b E_x^b \tag{314}$$

transmitted fields

$$\mathscr{H}_x^t = -E_y^t \qquad E_x^t = E_x^t \tag{315}$$

$$\mathscr{H}_y^t = E_x^t \qquad E_y^t = E_y^t \tag{316}$$

continuity of tangential fields at the interface After applying Snell's law, the system of equations is

$$
\begin{bmatrix} -1 & -1 & -1 & 0 \\ \rho_a & \rho_b & 0 & -1 \\ \mathfrak{n}_a\rho_a & \mathfrak{n}_b\rho_b & 0 & 1 \\ \mathfrak{n}_a & \mathfrak{n}_b & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E_x^a \\ E_x^b \\ E_x^t \\ E_y^t \end{bmatrix} = \begin{bmatrix} -E_x^i \\ -E_y^i \\ \mathfrak{n}_i E_y^i \\ -\mathfrak{n}_i E_x^i \end{bmatrix} \tag{317}
$$

The transmitted wave filed solutions are then

$$
\mathscr{H}_x^t = -\frac{-iE_x^i\left(\mathfrak{n}_a\mathfrak{n}_i\rho_b - \mathfrak{n}_a\rho_b + \mathfrak{n}_b\mathfrak{n}_i\rho_a - \mathfrak{n}_b\rho_a\right) + E_y^i\left(2i\mathfrak{n}_a\mathfrak{n}_b - \mathfrak{n}_a\mathfrak{n}_i\rho_b + i\mathfrak{n}_a + \mathfrak{n}_b\mathfrak{n}_i\rho_a + i\mathfrak{n}_b + \mathfrak{n}_i\rho_a - \mathfrak{n}_i\rho_b\right)}{2i\mathfrak{n}_a\mathfrak{n}_b - \mathfrak{n}_a\rho_b + i\mathfrak{n}_a + \mathfrak{n}_b\rho_a + i\mathfrak{n}_b + \rho_a - \rho_b} \tag{318}
$$

$$
\mathscr{H}_y^t = \frac{E_x^i\left(2i\mathfrak{n}_a\mathfrak{n}_b + i\mathfrak{n}_a\mathfrak{n}_i - \mathfrak{n}_a\rho_b + i\mathfrak{n}_b\mathfrak{n}_i + \mathfrak{n}_b\rho_a + \mathfrak{n}_i\rho_a - \mathfrak{n}_i\rho_b\right) + E_y^i\left(\mathfrak{n}_a\mathfrak{n}_i - \mathfrak{n}_a - \mathfrak{n}_b\mathfrak{n}_i + \mathfrak{n}_b\right)}{2i\mathfrak{n}_a\mathfrak{n}_b - \mathfrak{n}_a\rho_b + i\mathfrak{n}_a + \mathfrak{n}_b\rho_a + i\mathfrak{n}_b + \rho_a - \rho_b} \tag{319}
$$

$$
E_x^t = \frac{E_x^i\left(2i\mathfrak{n}_a\mathfrak{n}_b + i\mathfrak{n}_a\mathfrak{n}_i - \mathfrak{n}_a\rho_b + i\mathfrak{n}_b\mathfrak{n}_i + \mathfrak{n}_b\rho_a + \mathfrak{n}_i\rho_a - \mathfrak{n}_i\rho_b\right) + E_y^i\left(\mathfrak{n}_a\mathfrak{n}_i - \mathfrak{n}_a - \mathfrak{n}_b\mathfrak{n}_i + \mathfrak{n}_b\right)}{2i\mathfrak{n}_a\mathfrak{n}_b - \mathfrak{n}_a\rho_b + i\mathfrak{n}_a + \mathfrak{n}_b\rho_a + i\mathfrak{n}_b + \rho_a - \rho_b} \tag{320}
$$

$$
E_y^t = \frac{-iE_x^i\left(\mathfrak{n}_a\mathfrak{n}_i\rho_b - \mathfrak{n}_a\rho_b + \mathfrak{n}_b\mathfrak{n}_i\rho_a - \mathfrak{n}_b\rho_a\right) + E_y^i\left(2i\mathfrak{n}_a\mathfrak{n}_b - \mathfrak{n}_a\mathfrak{n}_i\rho_b + i\mathfrak{n}_a + \mathfrak{n}_b\mathfrak{n}_i\rho_a + i\mathfrak{n}_b + \mathfrak{n}_i\rho_a - \mathfrak{n}_i\rho_b\right)}{2i\mathfrak{n}_a\mathfrak{n}_b - \mathfrak{n}_a\rho_b + i\mathfrak{n}_a + \mathfrak{n}_b\rho_a + i\mathfrak{n}_b + \rho_a - \rho_b} \tag{321}
$$

8 conclusions

In this report, wave field polarization and mode contribution were calculated for a transionospheric EM wave. The ionosphere is treated as a magnetized plasma and assumed to have definite lower and upper boundaries. This construct allows for the solution of transmitted and reflected fields at each boundary. The process of solving the transmitted and reflected wave fields also gives the mode content of each spatial component of the wave fields.

References

[1] D. G. Swanson, *Plasma Waves.* Institute of Physics, second ed., 2003.

[2] T. H. Stix, *Waves in Plasmas.* Springer, second ed., 1992.

[3] F. F. Chen, *Plasma Physics and Controlled Fusion.* Plenum Press, second ed., 1988.

[4] K. G. Budden, *Radio Waves in the Ionosphere.* Cambridge University Press, 1966.

[5] K. G. Budden, *The Propagation of Radio Waves.* Cambridge University Press, 1985.

[6] J. D. Kraus, *Antennas.* McGraw Hill, second ed., 1988.

[7] J. D. Kraus, *Radio Astronomy.* Cygnus-Quasar, second ed., 1986.

[8] C. A. Balanis, *Antenna Theory: Analysis and Design.* John Wiley and Sons, second ed., 1997.

[9] K. C. Yeh and C. H. Liu, *Theory of Ionospheric Waves.* Academic Press, 1972.

[10] H. Booker, "Oblique propagation of electromagnetic waves in a slowly-varying non-isotropic medium," *Proc. R. Soc. London*, vol. 155, pp. 235–257, 1936.

[11] H. Booker, "Propagation of wave packets incident obliquely on a stratified doubly refracting ionosphere," *Phil. Trans. R. Soc. London*, vol. 237, pp. 411–451, 1939.

[12] H. Booker, "Application of the magnetoionic theory to radio waves incident obliquely upon a horizontally stratified ionosphere," *J. Geophys. Res.*, vol. 54, pp. 243–274, 1949.

[13] G. B. Arfken, *Mathematical Methods for Physicists.* Academic Press, third ed., 1985.

[14] A. W. Goodman and E. B. Saff, *Calculus: Concepts and Calculations.* Macmillan Publishing Co. Inc., 1981.

[15] M. Light, P. Colestock, L. Romero, T. Light, and D. Smith, "Bdv generation ii-f emp source-to-sensor model modules: Signal generation, signal propagation, noise, low and high band antennas," Technical Report LA-CP 13-01386, LANL.

[16] M. Light, "Ray trace modeling code," Technical Report LA-UR 19-26644, LANL.

[17] M. Light, "Ionospheric transfer function tests: Line of sight and slab," Technical Report DRAFT, LANL.